International Journal of Mathematics and Statistics Invention (IJMSI)
E-ISSN: 2321 – 4767 P-ISSN: 2321 - 4759
www.ijmsi.org Volume 2 Issue 8 || August. 2014 || PP-36-46

On optimal computational algorithm and transition cardinalities for solution matrices of single–delaylinear neutral scalar differential equations.

Ukwu Chukwunenye
Department of Mathematics University of Jos P.M.B 2084 Jos, Nigeria

ABSTRACT : *This paper established an optimal computationalalgorithm for solution matrices of single–delay autonomous linear neutral equations based on the established expressions of such matrices on a horizon of length equal to five times the delay. The development of the solution matrices exploited the continuity of these matrices for positive time periods, the method of steps, change of variablesand theory of linear difference equations to obtain these matrices on successive intervals of length equal to the delay h.*

KEYWORDS: *Algorithm, Equations, Matrices, Neutral, Solution.*

I. INTRODUCTION

Solution matrices are integral components of variation of constants formulas in the computations of solutions of linear and perturbed linear functional differential equations, Ukwu [1]. But quite curiously, no other author has made any serious attemptto investigate the existence or otherwise of their general expressions for various classes of these equations. Effort has usually focused on the single – delay model and the approach has been to start from the interval $[0,h]$, compute the solution matrices and solutions for given problem instances and then use the method of steps to extend these to the intervals $[kh,(k+1)h]$, for positive integral k, not exceeding 2, for the most part. Such approach is rather restrictive and doomed to failure in terms of structure for arbitrary k. In other words such approach fails to address the issue of the structure of solution matrices and solutions quite vital for real-world applications.With a view to addressing such short-comings, Ukwu and Garba[2] blazed the trail by consideringthe class of double – delay scalar differential equations:

$$\dot{x}(t) = ax(t) + bx(t-h) + cx(t-2h),\ t \in \mathbf{R},$$

where a, b and c are arbitrary real constants. By deploying ingenious combinations of summation notations, multinomial distribution, greatest integer functions, change of variables techniques, multiple integrals, as well as the method of steps, the paperderived the following optimal expressions for the solution matrices:

$$Y(t) = \begin{cases} e^{at},\ t \in J_0; \\ e^{at} + \sum_{i=1}^{k} b^i \frac{(t-ih)^i}{i!} e^{a(t-ih)} + \sum_{j=1}^{\left[\left[\frac{k}{2}\right]\right]} \sum_{i=0}^{k-2j} \frac{b^i c^j}{i!\,j!} \left(t-[i+2j]h\right)^{i+j} e^{a(t-[i+2j]h))};\ t \in J_k, k \geq 1 \end{cases}$$

where $J_k = [kh,(k+1)h], k \in \{0,1,\cdots\}$, $\left[\left[.\right]\right]$ denotes the greatest integer function, and $Y(t)$ denotes a generic solution matrix of the above class of equations for $t \in \mathbf{R}$. See also [3].

This article makes a positive contribution to knowledge bydevising a computational algorithmfor the solution matrices of linear neutral equations on $(-\infty,\infty)$.

II. RESULTS AND DISCUSSIONS

Observe that the above piece-wise expressions for $Y(t)$ may be restated more compactly in the form

$$Y(t) = \sum_{j=0}^{\left[\left[\frac{k}{2}\right]\right]} \sum_{i=0}^{k-2j} d_{ij}\left(t-[i+2j]h\right)^{i+j} e^{a(t-[i+2j]h))}\operatorname{sgn}\left(\max\left\{k+1,0\right\}\right)\ ;\ t \in J_k, k \geq 0$$

$$\text{where } d_{ij} = \frac{b^i c^j}{i!\,j!},\ j \in \left\{0,\cdots,\left[\left[\frac{k}{2}\right]\right]\right\},\ i \in \left\{0,\cdots,k-2j\right\}$$

Let $Y_{k-i}(t-ih)$ be a solution matrix of

$$\dot{x}(t) = a_{-1}\dot{x}(t-h) + a_0 x(t) + a_1 x(t-h), \qquad (1)$$

on the interval $J_{k-i} = [(k-i)h, (k+1-i)h], k \in \{0,1,\cdots\}, i \in \{0,1,\ 2\}$, where

$$Y(t) = \begin{cases} 1, t = 0 \\ 0, t < 0 \end{cases} \qquad (2)$$

Note that $Y(t)$ is a generic solution matrix for any $t \in \mathbf{R}$.

The coefficients $a_{-1}, a_0,\ a_1$ and the associated functions are all from the real domain.

The following theorem is preliminary to the devising/construction of an optimal computational algorithm for $Y(t)$.

2.1 Theorem on $Y(t)$

For $t \in J_k, k \in \left\{0,1,2,3,4,5\right\}$,

$$Y(t) = e^{a_0 t} + \sum_{i=1}^{k}\frac{\left(a_{-1}a_0 + a_1\right)^i}{i!}\left(t-ih\right)^i e^{a_0(t-ih)}\operatorname{sgn}\left(\max\left\{0,k\right\}\right)$$

$$+\sum_{i=1}^{k-1} a_{-1}^i\left(a_{-1}a_0+a_1\right)(t-[i+1]h)e^{a_0(t-[i+1]h)}\operatorname{sgn}\left(\max\left\{0,k-1\right\}\right)$$

$$+\ a_{-1}\left(a_{-1}a_0+a_1\right)^2\left(t-3h\right)^2 e^{a_0(t-3h)}\operatorname{sgn}\left(\max\left\{0,k-2\right\}\right)$$

$$+\left[a_{-1}\left(a_{-1}a_0+a_1\right)^3\frac{\left(t-4h\right)^3}{3}+\frac{3}{2}a_{-1}^2\left(a_{-1}a_0+a_1\right)^2\left(t-4h\right)^2\right]e^{a_0(t-4h)}\operatorname{sgn}\left(\max\left\{0,k-3\right\}\right)$$

$$+\left[\begin{array}{l} a_{-1}\left(a_{-1}a_0+a_1\right)^4\dfrac{\left(t-5h\right)^4}{8} \\ +a_{-1}^2\left(a_{-1}a_0+a_1\right)^3\left(t-5h\right)^2+2a_{-1}^3\left(a_{-1}a_0+a_1\right)^2\left(t-5h\right)^2 \end{array}\right]e^{a_0(t-5h)}\operatorname{sgn}\left(\max\left\{0,k-4\right\}\right)$$

Proof

On $\left(0,h\right)\subset J_0, Y(t-h)=0 \Rightarrow \dot{Y}(t)=a_0 Y(t)$ a.e. on $[0,h] \Rightarrow Y(t)\equiv Y_1(t)=e^{a_0 t}C; Y(0)=1 \Rightarrow C=1$

$$\Rightarrow Y(t)=e^{a_0 t} \text{ on } J_0=[0,h], \text{ as in the single - and double - delay systems.}$$

Consider the interval $(h,2h)$. Then

$$t-h\in[0,h]\Rightarrow Y(t-h)=e^{a_0(t-h)}\Rightarrow \dot{Y}(t-h)=A_0 e^{a_0(t-h)}\Rightarrow \dot{Y}(t)-a_0Y(t)=\left(a_{-1}a_0+a_1\right)e^{a_0(t-h)}$$

$$\Rightarrow\ \frac{d}{dt}\left[e^{-a_0 t}Y(t)\right]=e^{-a_0 t}\left[\dot{Y}(t)-a_0 Y(t)\right]=\left(a_{-1}a_0+a_1\right)e^{a_0(t-h)}$$

$$\Rightarrow e^{-a_0 t}Y(t)-e^{-a_0 h}Y(h)=\int_h^t e^{-a_0 s}\left(a_{-1}a_0+a_1\right)e^{a_0(s-h)}ds=\int_h^t e^{-a_0 h}\left(a_{-1}a_0+a_1\right)ds$$

$$\Rightarrow Y(t)=e^{a_0(t-h)}Y(h)+\left(a_{-1}a_0+a_1\right)(t-h)e^{-a_0(t-h)}=e^{a_0 t}+\left(a_{-1}a_0+a_1\right)(t-h)e^{a_0(t-h)},\text{ on } J_1.$$

$$t\in J_2\Rightarrow t-h\in J_1\Rightarrow Y(t-h)=e^{a_0(t-h)}+\left(a_{-1}a_0+a_1\right)(t-2h)e^{a_0(t-2h)},\text{ on } J_2.$$

On $J_k, k\geq 2,$ the relation $\frac{d}{dt}\left[e^{-a_0 t}Y(t)\right]=e^{-a_0 t}\left[a_1 Y(t-h)+a_{-1}\dot{Y}(t-h)\right]$ a.e.

$$\Rightarrow Y(t)=e^{a_0(t-kh)}Y(kh)+\int_{kh}^t e^{a_0(t-s)}\left[a_1 Y(s-h)+a_{-1}\dot{Y}(s-h)\right]ds \tag{3}$$

$$t\in J_2\Rightarrow t-h\in J_1, 2h\in J_1\Rightarrow Y(t-h)=e^{a_0(t-h)}+\left(a_{-1}a_0+a_1\right)(t-2h)e^{a_0(t-2h)},\text{ on } J_2.$$

$$Y(2h)=e^{a_0 2h}+\left(a_{-1}a_0+a_1\right)he^{a_0 h}$$

$$\Rightarrow Y(t)=e^{a_0(t-2h)}\left[e^{a_0 2h}+\left(a_{-1}a_0+a_1\right)he^{a_0 h}\right]+\int_{2h}^t e^{a_0(t-s)}a_1\left[e^{a_0(s-h)}+\left(a_{-1}a_0+a_1\right)(s-2h)e^{a_0(s-2h)}\right]ds$$

$$+\int_{2h}^t e^{a_0(t-s)}\left[a_{-1}\frac{d}{ds}\left[e^{a_0(s-h)}+\left(a_{-1}a_0+a_1\right)(s-2h)e^{a_0(s-2h)}\right]\right]ds$$

$$\Rightarrow Y(t)=e^{a_0 t}+\left(a_{-1}a_0+a_1\right)he^{a_0(t-h)}+(t-2h)a_1e^{a_0(t-h)}+a_1\int_{2h}^t\left(a_{-1}a_0+a_1\right)(s-2h)e^{a_0(t-2h)}ds$$

$$+a_{-1}a_0(t-2h)e^{a_0(t-h)}+a_{-1}\left(a_{-1}a_0+a_1\right)\left[(t-2h)+a_0\frac{\left(t-2h\right)^2}{2}\right]e^{a_0(t-2h)}$$

$$\Rightarrow Y(t)=e^{a_0 t}+\left[a_{-1}a_0+a_1\right](t-h)e^{a_0(t-h)}+\frac{\left(a_{-1}a_0+a_1\right)^2}{2}\left(t-2h\right)^2e^{a_0(t-2h)}$$

$$+a_{-1}\left(a_{-1}a_0+a_1\right)e^{a_0(t-2h)}\left(t-2h\right);\ t\in J_2.$$

Thus the theorem is established for $t\in J_k, k\in\{0, 1, 2\}$; needless to say that $\sum_{i=i_1}^{i_2}(.)=0, \text{ if } i_2<i_1.$

Note that the upper limit, $k-1$ in the second summation could be replaced explicitly by $\max\{1, k-1\}$.
Now consider the interval J_3; $s,t\in J_3\Rightarrow 3h\in J_2\cap J_3=\{3h\}$ and $s-h\in J_2$; hence

$$Y(3h)=e^{3a_0 h}+\left[a_{-1}a_0+a_1\right]2he^{2a_0 h}+\left(a_{-1}a_0+a_1\right)^2\frac{h^2}{2}e^{a_0 h}+a_{-1}\left(a_{-1}a_0+a_1\right)he^{a_0 h}$$

$$=e^{3a_0 h}+2\left[a_{-1}a_0+a_1\right]he^{2a_0 h}+\left(a_{-1}a_0+a_1\right)^2\frac{h^2}{2}e^{a_0 h}+a_{-1}\left(a_{-1}a_0+a_1\right)he^{a_0 h}$$

$$\Rightarrow Y(s-h)=e^{a_0(s-h)}+\left[a_{-1}a_0+a_1\right](s-2h)e^{a_0(s-2h)}+\frac{\left(a_{-1}a_0+a_1\right)^2}{2}\left(s-3h\right)^2e^{a_0(s-3h)}$$

$$+a_{-1}\left(a_{-1}a_0+a_1\right)e^{a_0(s-3h)}\left(s-3h\right);\ s\in J_3.$$

From the relation (3), we obtain

$$Y(t)=e^{a_0t}+2\left[a_{-1}a_0+a_1\right]he^{a_0(t-h)}+\left(a_{-1}a_0+a_1\right)^2\frac{h^2}{2}e^{a_0(t-2h)}+a_{-1}\left(a_{-1}a_0+a_1\right)he^{a_0(t-2h)}$$

$$+a_1\int_{3h}^{t}e^{a_0(t-s)}\begin{pmatrix}e^{a_0(s-h)}+\left[a_{-1}a_0+a_1\right](s-2h)e^{a_0(s-2h)}+\dfrac{\left(a_{-1}a_0+a_1\right)^2}{2}\left(s-3h\right)^2e^{a_0(s-3h)}\\+a_{-1}\left(a_{-1}a_0+a_1\right)\left(s-3h\right)e^{a_0(s-3h)}\end{pmatrix}ds$$

$$+a_{-1}\int_{3h}^{t}e^{a_0(t-s)}\frac{d}{ds}\begin{pmatrix}e^{a_0(s-h)}+\left[a_{-1}a_0+a_1\right](s-2h)e^{a_0(s-2h)}+\dfrac{\left(a_{-1}a_0+a_1\right)^2}{2}\left(s-3h\right)^2e^{a_0(s-3h)}\\+a_{-1}\left(a_{-1}a_0+a_1\right)\left(s-3h\right)e^{a_0(s-3h)}\end{pmatrix}ds$$

$$\Rightarrow Y(t)=e^{a_0t}+2\left[a_{-1}a_0+a_1\right]he^{a_0(t-h)}+\left(a_{-1}a_0+a_1\right)^2\frac{h^2}{2}e^{a_0(t-2h)}+a_{-1}\left(a_{-1}a_0+a_1\right)he^{a_0(t-2h)}$$

$$+a_1(t-3h)e^{a_0(t-h)}+a_1\left[a_{-1}a_0+a_1\right]\frac{(t-2h)^2e^{a_0(t-2h)}}{2}-a_1\left[a_{-1}a_0+a_1\right]\frac{h^2e^{a_0(t-2h)}}{2}$$

$$+a_1\frac{\left(a_{-1}a_0+a_1\right)^2}{3!}\left(t-3h\right)^3e^{a_0(t-3h)}+\frac{a_{-1}a_1\left(a_{-1}a_0+a_1\right)(t-3h)^2}{2}e^{a_0(t-3h)}$$

$$+a_{-1}a_0(t-3h)e^{a_0(t-h)}+a_{-1}\left[a_{-1}a_0+a_1\right](t-3h)e^{a_0(t-2h)}+\frac{a_{-1}a_0\left[a_{-1}a_0+a_1\right](t-2h)^2}{2}e^{a_0(t-2h)}$$

$$-\frac{a_{-1}a_0\left[a_{-1}a_0+a_1\right]h^2}{2}e^{a_0(t-2h)}+\frac{a_{-1}\left(a_{-1}a_0+a_1\right)^2}{2}\left[\left(t-3h\right)^2+a_0\frac{\left(t-3h\right)^3}{3}\right]e^{a_0(t-3h)}$$

$$+a_{-1}^2\left(a_{-1}a_0+a_1\right)\left[t-3h+a_0\frac{(t-3h)^2}{2!}\right]e^{a_0(t-3h)}$$

The evaluation of the integrals and skillful collection of like terms result in the following expression for $Y(t)$:

$$Y(t)=e^{a_0t}+\left[a_{-1}a_0+a_1\right](t-h)e^{a_0(t-h)}+\frac{\left(a_{-1}a_0+a_1\right)^2}{2!}\left(t-2h\right)^2e^{a_0(t-2h)}$$

$$+\frac{\left(a_{-1}a_0+a_1\right)^3}{3!}\left(t-3h\right)^3e^{a_0(t-3h)}+a_{-1}\left(a_{-1}a_0+a_1\right)\left(t-2h\right)e^{a_0(t-2h)}+a_{-1}^2\left(a_{-1}a_0+a_1\right)(t-3h)e^{a_0(t-3h)}$$

$$+a_{-1}\left(a_{-1}a_0+a_1\right)^2\left(t-3h\right)^2e^{a_0(t-3h)}$$

Observe that for $k\in\{0,1,2,3\}$ and $t\in J_k$,

$$Y(t)=e^{a_0t}+\left(\sum_{i=1}^{k}\frac{\left(a_{-1}a_0+a_1\right)^i}{i!}\left(t-ih\right)^ie^{a_0(t-ih)}\right)\operatorname{sgn}\left(\max\left\{0,k\right\}\right)$$

$$+\sum_{i=1}^{k-1}a_{-1}^i\left(a_{-1}a_0+a_1\right)(t-[i+1]h)e^{a_0(t-[i+1]h)}\operatorname{sgn}\left(\max\left\{0,k-1\right\}\right)$$

$$+a_{-1}\left(a_{-1}a_0+a_1\right)^2(t-3h)^2e^{a_0(t-3h)}\operatorname{sgn}\left(\max\left\{0,k-2\right\}\right)\qquad(4)$$

A definite pattern is yet to emerge; so the process continues.

Now consider the interval J_4; $s,t\in J_4\Rightarrow 4h\in J_3\cap J_4=\{4h\}$ and $s-h\in J_3$; hence the relation (3) implies that

$$Y(t)=e^{a_0(t-4h)}\left[\begin{array}{l} e^{4a_0h}+\sum_{i=1}^{3}\frac{\left(a_{-1}a_0+a_1\right)^i}{i!}\left([4-i]h\right)^i e^{a_0[4-i]h}+\sum_{i=1}^{2}a_{-1}^i\left(a_{-1}a_0+a_1\right)[3-i]he^{a_0([3-i]h)} \\ +\ a_{-1}\left(a_{-1}a_0+a_1\right)^2h^2e^{a_0h} \end{array}\right]$$

$$+\int_{4h}^{t}a_1e^{a_0(t-s_4)}\left[\begin{array}{l} e^{a_0(s_4-h)}+\left(\sum_{i=1}^{3}\frac{\left(a_{-1}a_0+a_1\right)^i}{i!}\left(s_4-[i+1]h\right)^i e^{a_0\left(s_4-[i+1]h\right)}\right) \\ +\sum_{i=1}^{2}a_{-1}^i\left(a_{-1}a_0+a_1\right)(s_4-[i+2]h)e^{a_0\left(s_4-[i+2]h\right)}+a_{-1}\left(a_{-1}a_0+a_1\right)^2(s_4-4h)^2e^{a_0(s_4-4h)} \end{array}\right]ds_4$$

$$+\int_{4h}^{t}a_{-1}e^{a_0(t-s_4)}\frac{d}{ds_4}\left[\begin{array}{l} e^{a_0(s_4-h)}+\left(\sum_{i=1}^{3}\frac{\left(a_{-1}a_0+a_1\right)^i}{i!}\left(s_4-[i+1]h\right)^i e^{a_0\left(s_4-[i+1]h\right)}\right) \\ +\sum_{i=1}^{2}a_{-1}^i\left(a_{-1}a_0+a_1\right)(s_4-[i+2]h)e^{a_0\left(s_4-[i+2]h\right)}+a_{-1}\left(a_{-1}a_0+a_1\right)^2(s_4-4h)^2e^{a_0(s_4-4h)} \end{array}\right]ds_4$$

$$\Rightarrow Y(t)=e^{a_0t}+\sum_{i=1}^{3}\frac{\left(a_{-1}a_0+a_1\right)^i}{i!}\left([4-i]h\right)^i e^{a_0(t-ih)}+\sum_{i=1}^{2}a_{-1}^i\left(a_{-1}a_0+a_1\right)[3-i]he^{a_0(t-[i+1]h)}$$
$$+\ a_{-1}\left(a_{-1}a_0+a_1\right)^2h^2e^{a_0(t-3h)}$$
$$+a_1\left(t-4h\right)e^{a_0(t-h)}+\sum_{i=1}^{3}\frac{a_1\left(a_{-1}a_0+a_1\right)^i}{(i+1)!}\left(t-[i+1]h\right)^{i+1}e^{a_0\left(t-[i+1]h\right)}$$
$$-\sum_{i=1}^{3}\frac{a_1\left(a_{-1}a_0+a_1\right)^i}{(i+1)!}\left([3-i]h\right)^{i+1}e^{a_0\left(t-[i+1]h\right)}+\sum_{i=1}^{2}a_1a_{-1}^i\left(a_{-1}a_0+a_1\right)\frac{(t-[i+2]h)^2e^{a_0(t-[i+2]h)}}{2}$$
$$-\sum_{i=1}^{2}a_1a_{-1}^i\left(a_{-1}a_0+a_1\right)\frac{([2-i]h)^2e^{a_0\left(t-[i+2]h\right)}}{2}+a_1a_{-1}\left(a_{-1}a_0+a_1\right)^2\frac{(t-4h)^3}{3}e^{a_0(t-4h)}$$
$$+a_{-1}a_0\left(t-4h\right)e^{a_0(t-h)}+\sum_{i=1}^{3}\frac{a_{-1}\left(a_{-1}a_0+a_1\right)^i}{i!}\left(t-[i+1]h\right)^i e^{a_0\left(t-[i+1]h\right)}-\sum_{i=1}^{3}\frac{a_{-1}\left(a_{-1}a_0+a_1\right)^i}{i!}\left([3-i]h\right)^i e^{a_0\left(t-[i+1]h\right)}$$
$$+\sum_{i=1}^{3}a_{-1}a_0\frac{\left(a_{-1}a_0+a_1\right)^i}{(i+1)!}\left(t-[i+1]h\right)^{i+1}e^{a_0\left(t-[i+1]h\right)}-\sum_{i=1}^{3}a_{-1}a_0\frac{\left(a_{-1}a_0+a_1\right)^i}{(i+1)!}\left([3-i]h\right)^{i+1}e^{a_0\left(t-[i+1]h\right)}$$
$$+\sum_{i=1}^{2}a_{-1}^{i+1}\left(a_{-1}a_0+a_1\right)(t-4h)e^{a_0\left(t-[i+2]h\right)}+\sum_{i=1}^{2}a_{-1}^{i+1}a_0\left(a_{-1}a_0+a_1\right)\frac{\left(t-[i+2]h\right)^2}{2}e^{a_0(t-[i+2]h)}$$
$$-\sum_{i=1}^{2}a_{-1}^{i+1}a_0\left(a_{-1}a_0+a_1\right)\frac{\left([2-i]h\right)^2}{2}e^{a_0(t-[i+2]h)}+a_{-1}^2\left(a_{-1}a_0+a_1\right)^2\left(t-4h\right)^2e^{a_0(t-4h)}$$
$$+a_{-1}^2a_0\left(a_{-1}a_0+a_1\right)^2\frac{\left(t-4h\right)^3}{3}e^{a_0(t-4h)}$$

It is evident from change of variables and grouping techniques that

$$e^{a_0 t}+\sum_{i=1}^{3}\frac{a_1\left(a_{-1}a_0+a_1\right)^i}{(i+1)!}\left(t-\left[i+1\right]h\right)^{i+1}e^{a_0\left(t-\left[i+1\right]h\right)}+\sum_{i=1}^{3}a_{-1}a_0\frac{\left(a_{-1}a_0+a_1\right)^i}{(i+1)!}\left(t-\left[i+1\right]h\right)^{i+1}e^{a_0\left(t-\left[i+1\right]h\right)}$$

$$+a_1\left(t-4h\right)e^{a_0(t-h)}+a_{-1}a_0\left(t-4h\right)e^{a_0(t-h)}+\sum_{i=1}^{3}\frac{\left(a_{-1}a_0+a_1\right)^i}{i!}\left(\left[4-i\right]h\right)^i e^{a_0(t-ih)}$$

$$-\sum_{i=1}^{3}\frac{a_1\left(a_{-1}a_0+a_1\right)^i}{(i+1)!}\left([3-i]h\right)^{i+1}e^{a_0(t-[i+1]h)}-\sum_{i=1}^{3}a_{-1}a_0\frac{\left(a_{-1}a_0+a_1\right)^i}{(i+1)!}\left(\left[3-i\right]h\right)^{i+1}e^{a_0\left(t-[i+1]h\right)}$$

$$=e^{a_0 t}+\sum_{i=1}^{4}\frac{\left(a_{-1}a_0+a_1\right)^i}{i!}\left(t-ih\right)^i e^{a_0\left(t-ih\right)}; \qquad (5)$$

$$\sum_{i=1}^{2}a_1a_{-1}^i\left(a_{-1}a_0+a_1\right)\frac{(t-[i+2]h)^2e^{a_0\left(t-[i+2]h\right)}}{2}+\sum_{i=1}^{2}a_{-1}^{i+1}a_0\left(a_{-1}a_0+a_1\right)\frac{\left(t-\left[i+2\right]h\right)^2}{2}e^{a_0(t-[i+2]h)}$$

$$=a_{-1}\left(a_{-1}a_0+a_1\right)^2\frac{\left(t-3h\right)^2}{2}e^{a_0\left(t-3h\right)}+a_{-1}^2\left(a_{-1}a_0+a_1\right)^2\frac{\left(t-4h\right)^2}{2}e^{a_0\left(t-4h\right)} \qquad (6)$$

$$=\sum_{i=1}^{2}a_{-1}^i\left(a_{-1}a_0+a_1\right)^2\frac{(t-[i+2]h)^2e^{a_0\left(t-[i+2]h\right)}}{2};$$

Also,

$$a_{-1}\left(a_{-1}a_0+a_1\right)^2h^2e^{a_0(t-3h)}-\sum_{i=1}^{2}a_1a_{-1}^i\left(a_{-1}a_0+a_1\right)\frac{([2-i]h)^2e^{a_0\left(t-\left[i+2\right]h\right)}}{2}$$

$$-\sum_{i=1}^{2}a_{-1}^{i+1}a_0\left(a_{-1}a_0+a_1\right)\frac{\left(\left[2-i\right]h\right)^2}{2}e^{a_0\left(t-[i+2]h\right)}=a_{-1}\left(a_{-1}a_0+a_1\right)^2\frac{h^2}{2}e^{a_0(t-3h)} \qquad (7)$$

Furthermore,

$$\sum_{i=1}^{2}a_{-1}^i\left(a_{-1}a_0+a_1\right)\left[3-i\right]he^{a_0\left(t-[i+1]h\right)}+a_1a_{-1}\left(a_{-1}a_0+a_1\right)^2\frac{(t-4h)^3}{3}e^{a_0(t-4h)}$$

$$+\sum_{i=1}^{2}a_{-1}^{i+1}\left(a_{-1}a_0+a_1\right)(t-4h)e^{a_0\left(t-[i+2]h\right)}+a_{-1}^2\left(a_{-1}a_0+a_1\right)^2\left(t-4h\right)^2e^{a_0(t-4h)}$$

$$+a_{-1}^2a_0\left(a_{-1}a_0+a_1\right)^2\frac{\left(t-4h\right)^3}{3}e^{a_0(t-4h)}+\sum_{i=1}^{3}\frac{a_{-1}\left(a_{-1}a_0+a_1\right)^i}{i!}\left(t-\left[i+1\right]h\right)^i e^{a_0\left(t-\left[i+1\right]h\right)}$$

$$-\sum_{i=1}^{3}\frac{a_{-1}\left(a_{-1}a_0+a_1\right)^i}{i!}\left(\left[3-i\right]h\right)^i e^{a_0\left(t-\left[i+1\right]h\right)}$$

$$=\sum_{i=1}^{4-1}a_{-1}^i\left(a_{-1}a_0+a_1\right)\left(t-\left[i+1\right]h\right)e^{a_0\left(t-[i+1]h\right)}+\frac{1}{2}a_{-1}\left(a_{-1}a_0+a_1\right)^2\left(t-3h\right)^2e^{a_0(t-3h)}$$

$$+a_{-1}^2\left(a_{-1}a_0+a_1\right)^2\left(t-4h\right)^2e^{a_0(t-4h)}+a_{-1}\left(a_{-1}a_0+a_1\right)^3\frac{(t-4h)^3}{2}e^{a_0(t-4h)}$$

$$-a_{-1}\left(a_{-1}a_0+a_1\right)^2\frac{h^2}{2}e^{a_0(t-3h)} \qquad (8)$$

Adding up expressions (5), (6), (7) and (8) yields

$$Y(t) = e^{a_0 t} + \left(\sum_{i=1}^{4} \frac{(a_{-1}a_0 + a_1)^i}{i!} (t - ih)^i e^{a_0(t-ih)} \right)$$

$$+ \sum_{i=1}^{4-1} a_{-1}^i (a_{-1}a_0 + a_1)(t - [i+1]h) e^{a_0(t-[i+1]h)} + a_{-1}(a_{-1}a_0 + a_1)^2 (t-3h)^2 e^{a_0(t-3h)}$$

$$+ \left[a_{-1}(a_{-1}a_0 + a_1)^3 \frac{(t-4h)^3}{2} + \frac{3}{2} a_{-1}^2 (a_{-1}a_0 + a_1)^2 (t-4h)^2 \right] e^{a_0(t-4h)} \quad (9)$$

Hence for $t \in J_k, k \in \{0, 1, 2, 3, 4\}$,

$$Y(t) = e^{a_0 t} + \left(\sum_{i=1}^{k} \frac{(a_{-1}a_0 + a_1)^i}{i!} (t - ih)^i e^{a_0(t-ih)} \right) \operatorname{sgn}(\max\{0, k\})$$

$$+ \sum_{i=1}^{k-1} a_{-1}^i (a_{-1}a_0 + a_1)(t - [i+1]h) e^{a_0(t-[i+1]h)} \operatorname{sgn}(\max\{0, k-1\})$$

$$+ a_{-1}(a_{-1}a_0 + a_1)^2 (t-3h)^2 e^{a_0(t-3h)} \operatorname{sgn}(\max\{0, k-2\})$$

$$+ \left[a_{-1}(a_{-1}a_0 + a_1)^3 \frac{(t-4h)^3}{2} + \frac{3}{2} a_{-1}^2 (a_{-1}a_0 + a_1)^2 (t-4h)^2 \right] e^{a_0(t-4h)} \operatorname{sgn}(\max\{0, k-3\}) \quad (10)$$

Observe that for $t \in J_k$, $c_{0j} = \frac{1}{j!}$, $j \in \{1, 2, \cdots, k\}$; and $c_{i1} = 1$, $i \in \{1, 2, \cdots, k-1\}$. The transformation from $Y_k(t)$ to $Y_{k+1}(t)$ requires only the computations of c_{ij}, for $i \in \{1, 2, \cdots, k+1-2\}$, $j \in \{2, 3, \cdots, k+1-i\}$, $k \geq 3$, such that $i + j = k+1$, since c_{i1} is already known for each *i*. Therefore one need only determine $k-1$ new c_{ij} values, namely $c_{1k}, c_{2\,k-1}, c_{3\,k-2}, \cdots, c_{k-1\,2}$.

On a positive note, the author has successfully devised an optimal computational algorithm for the solution matrices without recourse to the class of differential equations (1) and expression (3), using $Y_1(t)$ as a starting point. This is the focus of the next result.

3. Main Result: A computational Algorithm for transiting from $Y_k(t)$ to $Y_{k+1}(t)$, $k \geq 1$

Let $t \in J_k$, let $\lambda_1, \lambda_2 \in \{0,1\}$. Suppose that $a_{-1}(a_{-1}a_0 + a_1) \neq 0$.

$$Y(t) = Y_{k+1}(t) = Y_1(t) + \sum_{\lambda_1+\lambda_2=1} \sum_{j=1}^{k} a_{-1}^{\lambda_1} \frac{(a_{-1}a_0 + a_1)^{j+\lambda_2}}{(j+\lambda_2)!} (t - [j+1]h)^{j+\lambda_2} e^{a_0(t-[j+1]h)}$$

$$+ \sum_{\lambda_1+\lambda_2=1} \sum_{i=1}^{k-1} a_{-1}^{i+\lambda_1} \frac{(a_{-1}a_0 + a_1)^{1+\lambda_2}}{1+\lambda_2} (t - [i+1]h)^{1+\lambda_2} e^{a_0(t-[i+2]h)} \operatorname{sgn}(\max\{0, k-1\})$$

$$+ \sum_{\lambda_1+\lambda_2=1} \sum_{i=1}^{k-2} \sum_{j=2}^{k-i} c_{ij} a_{-1}^{i+\lambda_1} \frac{(a_{-1}a_0 + a_1)^{j+\lambda_2}}{1+\lambda_2+(j-1)\operatorname{sgn}(\lambda_2)} (t - [i+j+1]h)^{j+\lambda_2} e^{a_0(t-[i+j+1]h)} \operatorname{sgn}(\max\{0, k-2\}) \quad (11)$$

for some real positive constants c_{ij} secured from $Y_k(t)$, with the process initiated at $k = 1$.

3.1 Remarks on the optimal computational algorithm

Observe that for $t \in J_k, Y_{k+1}(t)$ can be expressed in the equivalent form

$$Y_{k+1}(t) = Y_1(t)$$
$$+ \sum_{\lambda_1+\lambda_2=1} \sum_{i=0}^{k-1} \sum_{j=1}^{k-i} c_{ij} a_{-1}^{i+\lambda_1} \frac{(a_{-1}a_0 + a_1)^{j+\lambda_2}}{1+ j\,\mathrm{sgn}(\lambda_2)} (t-[i+j+1]h)^{j+\lambda_2} e^{a_0(t-[i+j+1]h)} \mathrm{sgn}\left(\max\{0, k-1\}\right) \quad (12)$$

for some real positive constants c_{ij} secured from $Y_k(t)$, with the process initiated at $k=1$.

Moreover $c_{0j} = \frac{1}{j!}, j \in \{1, 2, \cdots, k+1\}$; $c_{i1} = 1, i \in \{1, 2, \cdots, k\}$ and the transformation from $Y_k(t)$ to $Y_{k+1}(t)$ requires only the computations of c_{ij} for $i \in \{1, 2, \cdots, k+1-2\}$, $j \in \{2, 3, \cdots, k+1-i\}$, $k \geq 3$, such that $i + j = k+1$. Therefore one need only determine $= k-1$ new c_{ij} values, namely $c_{1k}, c_{2\,k-1}, c_{3\,k-2}, \cdots, c_{k-1\,2}$.

3.2 Interpretation of the computational algorithm for Y(t)

Stage 1:Transiting from $Y_k(t)$ to $Y_{k+1} - Y_1(t), k \in \{1, 2, 3, \cdots\}$

Perform the following operations on each term of $\left(Y_k(t) - e^{a_0 t}\right)$:

(i) Increment each power of a_{-1} by 1; preserve the power of $(a_{-1}a_0 + a_1)$

(ii) Let $(t-[.]h)^{\text{power of }(a_{-1}a_0+a_1)} \to (t-h-[.]h)^{\text{power of }(a_{-1}a_0+a_1)}$; $e^{a_0(t-[.]h)} \to e^{a_0(t-h-[.]h)}$

The operations (i) and (ii) yield exactly the same number of terms as in $\left(Y_k(t) - e^{a_0 t}\right)$

(iii) Increment each exponent of $(a_{-1}a_0 + a_1)$ by 1; preserve the exponent of a_{-1}

(iv) Let $(t-[.]h)^{\text{exponent of}(a_{-1}a_0+a_1)} \to (t-h-[.]h)^{1+\text{ old exponent of}(a_{-1}a_0+a_1)}$; $e^{a_0(t-[.]h)} \to e^{a_0(t-h-[.]h)}$

(v) Divide each term by the new exponent of $(a_{-1}a_0 + a_1)$ resulting from operation (iii),
where new exponent of $(a_{-1}a_0 + a_1) = 1 +$ old (preceding) exponent of $(a_{-1}a_0 + a_1)$, that is
new exponent of $(a_{-1}a_0 + a_1)$ = exponent of $(a_{-1}a_0 + a_1)$ from the resulting term in $Y_{k+1}(t)$
$= 1 +$ exponent of $(a_{-1}a_0 + a_1)$ from the term operated on, in $Y_k(t)$

The operations (iii), (iv) and (v) yield exactly the same number of terms as in $\left(Y_k(t) - e^{a_0 t}\right)$

(vi) Aggregate all terms resulting from operations (i) to (v) through appropriate groupings
of common factors of powers of $[a_{-1}$ and $(a_{-1}a_0 + a_1)]$.

Stage 2: Transiting from $(Y_{k+1} - Y_1(t))$ to $Y_{k+1}(t), k \in \{1, 2, 3, \cdots\}$

Secure $Y_{k+1}(t)$ by adding $Y_1(t)$ to the aggregated terms in (vi); in order words

$$Y_{k+1}(t) = Y_1(t) + \begin{pmatrix} \text{the resultant expressions from the application of the algorithm to} \\ \left(Y_k(t) - e^{a_0 t}\right); k \in \{1, 2, 3, \cdots\} \end{pmatrix}$$

3.3 Cardinality and CardinalityTransition Analyses on $Y_k(t)$

Denote the cardinality of $(.)$ by $|(.)|$. Then $|Y_{k+1}(t)| - |Y_k(t)| = k+1$ is the resulting autonomous nonhomogeneous linear difference equation, for $k \in \{1, 2, 3, \cdots\}$. Hence

$$|Y_{k+1}(t)| = 1 + \frac{(k+1)(k+2)}{2}, \; k \in \{0, 2, 3, \cdots\}, \text{noting that } |Y_1(t)| = 2.$$

Furthermore, for $k \in \{1, 2, \cdots\}$, the number of terms that need to be aggregated from $\left(Y_k(t) - e^{a_0 t}\right)$ to secure $\left(Y_{k+1}(t) - Y_1(t)\right)$ is $k(k+1)$, derived from the fact that there are $2\left(\left|Y_k(t) - \left|e^{a_0 t}\right|\right|\right)$ $= 2\left(\left|Y_k(t) - 1\right|\right)$ such terms. Therefore $k^2 + k + 2$ terms must be aggregated in the transition from $Y_k(t)$ to $Y_{k+1}(t)$.

3.4 Verification and illustrations of Algorithm 3

$$t \in J_2 \Rightarrow Y_2(t) = Y_1(t) + a_{-1}(a_{-1}a_0 + a_1)([t-h]-h)e^{a_0([t-h]-h)} + \frac{(a_{-1}a_0 + a_1)^{1+1}([t-h]-h)e^{a_0([t-h]-h)}}{1+1}$$

$$Y_2(t) = e^{a_0 t} + (a_{-1}a_0 + a_1)(t-h)e^{a_0(t-h)} + a_{-1}(a_{-1}a_0 + a_1)([t-h]-h)e^{a_0([t-h]-h)} + \frac{(a_{-1}a_0 + a_1)^{1+1}([t-h]-h)^{1+1}e^{a_0([t-h]-h)}}{1+1}$$

$$\Rightarrow Y(t) = e^{a_0 t} + \sum_{i=1}^{2} \frac{(a_{-1}a_0 + a_1)^{1+1}(t-ih)^i e^{a_0([t-h]-h)}}{i!} + a_{-1}\left(a_{-1}a_0 + a_1\right)(t-2h)e^{a_0(t-2h)}, t \in J_2.$$

It is straight-forward and easy to check that the algorithm verifies the rest of the computations for $Y(t)$, for $t \in J_3 \cup J_4 = [3h, 5h]$.

Finally, we apply the algorithm to extend the solution matrices to the interval $J_5 = [5h, 6h]$.
To achieve this, set $k = 4$, so that $k+1 = 5$; so we need only obtain the $k-1 = 4-1 = 3$ new coefficients c_{ij}, for $i \in \{1, 2, 3\}$, $j \in \{2, \cdots, 5-i\} : i + j = k+1$, namely c_{14}, c_{23} and c_{32}.
The application of the algorithm from J_4 to J_5 yields the following expression for $Y(t), t \in J_5$:

$$Y(t) = Y_1(t) + \left(\sum_{j=1}^{4} a_{-1} \frac{\left(a_{-1}a_0 + a_1\right)^j}{j!} \left(t - [j+1]h\right)^j e^{a_0\left(t-[j+1]h\right)} \right)$$
$$+ \sum_{i=1}^{4-1} a_{-1}^{i+1}\left(a_{-1}a_0 + a_1\right)(t-[i+1]h)e^{a_0\left(t-[i+1]h\right)} + a_{-1}^2\left(a_{-1}a_0 + a_1\right)^2 (t-4h)^2 e^{a_0(t-4h)}$$
$$+ \left[a_{-1}^2\left(a_{-1}a_0 + a_1\right)^3 \frac{(t-5h)^3}{2} + \frac{3}{2} a_{-1}^3\left(a_{-1}a_0 + a_1\right)^2 (t-5h)^2 \right] e^{a_0(t-5h)}$$
$$+ \left(\sum_{j=1}^{4} \frac{\left(a_{-1}a_0 + a_1\right)^{j+1}}{(j+1)\, j!} \left(t - [j+1]h\right)^j e^{a_0\left(t-[j+1]h\right)} \right)$$
$$+ \sum_{i=1}^{4-1} a_{-1}^i \frac{\left(a_{-1}a_0 + a_1\right)^2}{2} (t-[i+2]h)e^{a_0\left(t-[i+1]h\right)} + a_{-1} \frac{\left(a_{-1}a_0 + a_1\right)^3}{3} (t-4h)^2 e^{a_0(t-4h)}$$
$$+ \left[a_{-1}\left(a_{-1}a_0 + a_1\right)^4 \frac{(t-4h)^4}{2(4)} + \frac{3}{2(3)} a_{-1}^2\left(a_{-1}a_0 + a_1\right)^3 (t-5h)^2 \right] e^{a_0(t-5h)}$$

Aggregation of like terms: $Y_1(t) +$ the terms with $1 \le i + j \le 4$, together with $c_{05} = \frac{1}{5!}, c_{41} = 1$, evaluate to

$$e^{a_0 t}+\left(\sum_{j=1}^{5}\frac{\left(a_{-1}a_0+a_1\right)^j}{j!}\left(t-jh\right)^j e^{a_0\left(t-jh\right)}\right)+\sum_{i=1}^{5-1}a_{-1}^i\left(a_{-1}a_0+a_1\right)(t-[i+1]h)e^{a_0\left(t-[i+1]h\right)}$$

$$+a_{-1}\left(a_{-1}a_0+a_1\right)^2(t-3h)^2e^{a_0(t-3h)}+\left[a_{-1}\left(a_{-1}a_0+a_1\right)^3\frac{(t-4h)^3}{2}+\frac{3}{2}a_{-1}^2\left(a_{-1}a_0+a_1\right)^2(t-4h)^2\right]e^{a_0(t-4h)}$$

It follows from the aggregated terms that the values of the remaining three coefficients c_{14}, c_{23} and c_{23} are $\frac{1}{4!}+\frac{1}{8}$, $\frac{1}{2}+\frac{1}{2}$ and $\frac{3}{2}+\frac{1}{2}$ respectively. Therefore $c_{14}=\frac{1}{6}, c_{23}=1$ and $c_{32}=2.$ Hence

$$Y(t)=e^{a_0 t}+\left(\sum_{j=1}^{5}\frac{\left(a_{-1}a_0+a_1\right)^j}{j!}\left(t-jh\right)^j e^{a_0\left(t-jh\right)}\right)+\sum_{i=1}^{5-1}a_{-1}^i\left(a_{-1}a_0+a_1\right)(t-[i+1]h)e^{a_0\left(t-[i+1]h\right)}$$

$$+a_{-1}\left(a_{-1}a_0+a_1\right)^2(t-3h)^2e^{a_0(t-3h)}+\left[a_{-1}\left(a_{-1}a_0+a_1\right)^3\frac{(t-4h)^3}{2}+\frac{3}{2}a_{-1}^2\left(a_{-1}a_0+a_1\right)^2(t-4h)^2\right]e^{a_0(t-4h)}$$

$$+\left[a_{-1}\left(a_{-1}a_0+a_1\right)^4\frac{(t-4h)^4}{3!}+a_{-1}^2\left(a_{-1}a_0+a_1\right)^3(t-5h)^3+2a_{-1}^3\left(a_{-1}a_0+a_1\right)^2(t-5h)^2\right]e^{a_0(t-5h)};\ t\in J_5.$$

Moreover the general expresion for $Y(t), t\in\bigcup_{k=0}^{5}J_k$ can be stated as follows:

$$Y(t)=e^{a_0 t}+\left(\sum_{j=1}^{5}\frac{\left(a_{-1}a_0+a_1\right)^j}{j!}\left(t-jh\right)^j e^{a_0\left(t-jh\right)}\right)\max\left\{k,0\right\}$$

$$+\sum_{i=1}^{5-1}a_{-1}^i\left(a_{-1}a_0+a_1\right)(t-[i+1]h)e^{a_0\left(t-[i+1]h\right)}\max\left\{k-1,0\right\}$$

$$+a_{-1}\left(a_{-1}a_0+a_1\right)^2(t-3h)^2e^{a_0(t-3h)}\max\left\{k-2,0\right\}$$

$$+\left[a_{-1}\left(a_{-1}a_0+a_1\right)^3\frac{(t-4h)^3}{2}+\frac{3}{2}a_{-1}^2\left(a_{-1}a_0+a_1\right)^2(t-4h)^2\right]e^{a_0(t-4h)}\max\left\{k-3,0\right\}$$

$$+\left[\begin{array}{l}a_{-1}\left(a_{-1}a_0+a_1\right)^4\frac{(t-4h)^4}{3!}+a_{-1}^2\left(a_{-1}a_0+a_1\right)^3(t-5h)^3\\+2a_{-1}^3\left(a_{-1}a_0+a_1\right)^2(t-5h)^2\end{array}\right]e^{a_0(t-5h)}\max\left\{k-4,0\right\}.$$

IV. CONCLUSION

This article obtained an optimal computational schemefor the structure of the solution matrices of single-delay linear neutral differential equations by leveraging on the established expressions for such matrices on the time interval of length equal to five times the delay, starting from time zero. The scheme is iteratively based on transitions fromone time interval of length equal to the delay to the next contiguous intervalof length h, with the coefficients from the preceding interval preserved, two new coefficients updatedand the rest obtained from the aggregation of the components resulting from the afore-mentioned transitions. This algorithm alleviates the computational burden associated with relying on the equation (1) and the expression (3) fraught with proneness to computational errors and resolves to a great extent the lack of a general expression for the solution matrices.The structure of the algorithm is so simple that the solution matrix transitions from one interval to the next contiguous interval can be obtained by inspection and addition of two terms for each new coefficient.

REFERENCES

[1] Ukwu, C. (1987). Compactness of cores of targets for linear delay systems., J. Math. Analy.and Appl., Vol. 125, No. 2, August 1. pp. 323-330.
[2] Ukwu, C. and Garba, E.J.D. (2014w).Construction of optimal expressions for transition matrices of a class of double – delay scalar differential equations. *African Journal of Natural Sciences (AJNS). Vol. 16, 2014.*
[3] Ukwu, C. and E. J. D. Garba (2014e). Derivation of an optimal expression for solution matrices of a class of single-delay scalar differential equations. *Journal of Nigerian Association of Mathematical Physics*. Vol. ***26**, March 2014*.