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ALMOST KÄHLER EIGHT-DIMENSIONAL WALKER MANIFOLD

Abdoul Salam Diallo[1][2], Silas Longwap[3] and Fortuné Massamba[4]

Abstract. A Walker n-manifold is a pseudo-Riemannian manifold which admits a field of parallel null r-planes, with $r \leq \frac{n}{2}$. The canonical forms of the metrics were investigated by A. G. Walker [13]. Of special interest are the even-dimensional Walker manifolds ($n = 2m$) with fields of parallel null planes of half dimension ($r = m$). In this paper, we investigate geometric properties of some curvature tensors of an eight-dimensional Walker manifold. Theorems for the metric to be Einstein, locally conformally flat and for the Walker eight-manifold to admit a Kähler structure are given.

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1. Introduction

It is known that Walker metrics have served as a powerful tool for constructing interesting indefinite metrics which exhibit various aspects of geometric properties not given by any positive definite metrics. Among these, the significant Walker manifolds are the examples of the non-symmetric and non-homogeneous Osserman manifolds [2]. It was shown in [6, 9, 10] that the Walker 4-manifolds of neutral signature admit a pair comprising of an almost complex structure and an opposite almost complex structure, and that Petean's non-flat indefinite Kähler-Einstein metric on a torus was obtained as an example of Walker 4-manifolds. Banyaga and Massamba in [1] derived a Walker metric when studying the non-existence of certain Einstein metrics on some symplectic manifolds. Moreover, an indefinite Ricci flat strictly almost Kähler metric on eight-dimensional torus was reported in [11]. Thus the Walker 4- and 8-manifolds display a large variety of indefinite geometry in dimensions four and eight.

[1]School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Private Bag X01, Scottsville 3209, South Africa, and Université Alioune Diop de Bambey, UFR SATIC, Département de Mathématiques, B. P. 30, Bambey, Sénégal, e-mail: abdoulsalam.diallo@uadb.edu.sn, Diallo@ukzn.ac.za

[2]Corresponding author

[3]School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Private Bag X01, Scottsville 3209, South Africa, e-mail: longwap4all@yahoo.com

[4]School of Mathematics, Statistics and Computer Science, University of KwaZulu-Natal, Private Bag X01, Scottsville 3209, South Africa,
e-mail: massfort@yahoo.frMassamba@ukzn.ac.za

Our aim is to study restricted 8-Walker metrics by focusing on their curvature properties. The main results of this paper are the characterization of Walker metrics which are Einstein, locally conformally flat and Kähler. The paper is organized as follows. In Section 2, we recall some basic notions about Walker metrics. Two specific Walker metrics of 8-dimensional manifolds are investigated in Section 3. We find the form of the defining functions that makes those metrics similar to Einstein and locally conformally flat metrics. In the last section we give conditions for the Walker 8-manifold to admit a Kähler structure.

2. The canonical form of Walker metrics

Let M be a pseudo-Riemannian manifold of signature (n, n). Suppose that the tangent bundle TM splits as a sum of smooth sub-bundles $\mathcal{D}_1$ and $\mathcal{D}_2$, called distributions:

$$TM = \mathcal{D}_1 \oplus \mathcal{D}_2.$$

This define two complementary projections π_1 and π_2 of TM onto $\mathcal{D}_1$ and $\mathcal{D}_2$. We say that $\mathcal{D}_1$ is parallel distribution if $\nabla \pi_1 = 0$. Equivalently this means that if X_1 is any smooth vector field taking values in $\mathcal{D}_1$, then ∇X_1 again takes values in $\mathcal{D}_1$. If M is Riemannian, we can take $\mathcal{D}_2 = \mathcal{D}_1^\perp$ to be the orthogonal complement of $\mathcal{D}_1$ and in that case $\mathcal{D}_2$ is again parallel. In the pseudo-Riemannian setting, $\mathcal{D}_1 \cap \mathcal{D}_2$ need not be trivial. We say that $\mathcal{D}_1$ is a null parallel distribution if it is parallel and the metric restricted to $\mathcal{D}_1$ vanishes identically.

Walker [13] studied pseudo-Riemannian manifolds (M, g) with a parallel field of null planes $\mathcal{D}$ and derived a canonical form. Motivated by this seminal work, one says that a pseudo-Riemannian manifold M which admits a null parallel (i.e., degenerate) distribution $\mathcal{D}$ is a *Walker manifold*.

Canonical forms were known previously for parallel non-degenerate distributions. In this case, the metric tensor, in matrix notation, is expressed in the canonical form as

$$(g_{ij}) = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \tag{2.1}$$

where A is a symmetric $(r \times r)$- matrix whose coefficients are functions of $(u_1, \ldots, u_r)$ and B is a symmetric $(n-r) \times (n-r)$ matrix whose coefficients are functions of $(u_{r+1}, \ldots, u_n)$. Here n is the dimension of M and r is the dimension of the distribution $\mathcal{D}$. We will refer to [2] for the proof of the following theorems.

Theorem 2.1. *[2] A canonical form for an n-dimensional pseudo-Riemannian manifold (M, g) admitting a parallel field of null r-dimensional planes $\mathcal{D}$ is given by the metric tensor in matrix form as*

$$(g_{ij}) = \begin{pmatrix} 0 & 0 & \mathrm{Id}_r \\ 0 & A & H \\ \mathrm{Id}_r & {}^tH & B \end{pmatrix}, \tag{2.2}$$

where Id_r *is the* $(r \times r)$*-identity matrix and* A, B, H *are matrices whose coefficients are functions of the coordinates satisfying the following:*

1. A *and* B *are symmetric matrices of orders* $(n-2r)\times(n-2r)$ *and* $(r \times r)$*, respectively.* H *is a matrix of order* $(n-2r)\times r$ *and* ${}^tH$ *stands for the transpose of* H*.*

2. A *and* H *are independent of the coordinates* $(u_1, \ldots, u_r)$*.*

Furthermore, the null parallel r*-plane* $\mathcal{D}$ *is locally generated by the coordinate vector fields* $\{\partial_{u_1}, \ldots, \partial_{u_r}\}$*.*

Theorem 2.2. *[2] A canonical form for an* n*-dimensional pseudo-Riemannian manifold* (M, g) *admitting a strictly parallel field of null* r*-dimensional planes* $\mathcal{D}$ *is given by the metric tensor as in Theorem 2.1, where* B *is independent of the coordinates* $(u_1, \ldots, u_r)$*.*

Recall that a Walker metric is said to be *Einstein Walker metric* if its Ricci tensor is a scalar multiple of the metric at each point. Four-dimensional Einstein Walker manifolds form the underling structure of many geometric and physical models such as; *hh*-space in general relativity, *pp*-wave model and other areas, see, for example, [3] and references therein.

3. On eight-dimensional Walker metrics

A neutral g on an 8-manifold M is said to be a Walker metric if there exists a 4-dimensional null distribution $\mathcal{D}$ on M which is parallel with respect to g. From Walker theorem [13], there is a system of coordinates $(u_1, \ldots, u_8)$ with respect to which g takes the local canonical form

$$(g_{ij}) = \begin{pmatrix} 0 & \mathrm{Id}_4 \\ \mathrm{Id}_4 & B \end{pmatrix}, \tag{3.1}$$

where Id_4 is the (4×4)-identity matrix and B is an (4×4) symmetric matrix whose coefficients are the functions of $(u_1, \ldots, u_8)$. Note that g is of neutral signature $(4, 4)$ and that the parallel null 4-plane $\mathcal{D}$ is spanned locally by $\{\partial_1, \ldots, \partial_4\}$, where $\partial_i = \frac{\partial}{\partial_i}, i = 1, 2, 3, 4$.
In this paper, we consider the specific Walker metrics on 8-dimensional manifold M with B of the form

$$B = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix}, \tag{3.2}$$

where a is a function of the $(u_1, \ldots, u_8)$. We will denote by $a_i = \frac{\partial a(u_1, \ldots, u_8)}{\partial u_i}$. The non-vanishing components of the Christoffel symbols Γ_{ij}^k of the Levi-Civita connection of the Walker metric (3.1) and (3.2) are given by

(3.3)

$$
\begin{aligned}
\Gamma^1_{ij} &= \frac{-a}{2}\Big(\partial_i g_{j1} + \partial_j g_{i1} - \partial_1 g_{ij}\Big) + \frac{1}{2}\Big(\partial_i g_{j5} + \partial_j g_{i5} - \partial_5 g_{ij}\Big),\\
\Gamma^2_{ij} &= \frac{-a}{2}\Big(\partial_i g_{j2} + \partial_j g_{i2} - \partial_2 g_{ij}\Big) + \frac{1}{2}\Big(\partial_i g_{j6} + \partial_j g_{i6} - \partial_6 g_{ij}\Big),\\
\Gamma^3_{ij} &= \frac{-a}{2}\Big(\partial_i g_{j3} + \partial_j g_{i3} - \partial_3 g_{ij}\Big) + \frac{1}{2}\Big(\partial_i g_{j7} + \partial_j g_{i7} - \partial_7 g_{ij}\Big),\\
\Gamma^4_{ij} &= \frac{-a}{2}\Big(\partial_i g_{j4} + \partial_j g_{i4} - \partial_4 g_{ij}\Big) + \frac{1}{2}\Big(\partial_i g_{j8} + \partial_j g_{i8} - \partial_8 g_{ij}\Big),\\
\Gamma^5_{ij} &= \frac{1}{2}\Big(\partial_i g_{j1} + \partial_j g_{i1} - \partial_1 g_{ij}\Big), \quad \Gamma^6_{ij} = \frac{1}{2}\Big(\partial_i g_{j2} + \partial_j g_{i2} - \partial_2 g_{ij}\Big),\\
\Gamma^7_{ij} &= \frac{1}{2}\Big(\partial_i g_{j3} + \partial_j g_{i3} - \partial_3 g_{ij}\Big), \quad \Gamma^8_{ij} = \frac{1}{2}\Big(\partial_i g_{j4} + \partial_j g_{i4} - \partial_4 g_{ij}\Big).
\end{aligned}
$$

A straightforward calculation shows that the non-vanishing components of Levi-Civita connection of a Walker metric (3.1) and (3.2) are given by

(3.4)

$$
\begin{aligned}
\nabla_{\partial_5}\partial_5 &= \frac{1}{2}\Big((aa_1 + a_5)\partial_1 + (aa_2 - a_6)\partial_2 + (aa_3 - a_7)\partial_3 + (aa_4 - a_8)\partial_4\Big)\\
&\quad -\frac{1}{2}\big(a_1\partial_5 + a_2\partial_6 + a_3\partial_7 + a_4\partial_8\big),\\
\nabla_{\partial_6}\partial_6 &= \frac{1}{2}\Big((aa_1 - a_5)\partial_1 + (aa_2 + a_6)\partial_2 + (aa_3 - a_7)\partial_3 + (aa_4 - a_8)\partial_4\Big)\\
&\quad -\frac{1}{2}\big(a_1\partial_5 + a_2\partial_6 + a_3\partial_7 + a_4\partial_8\big),\\
\nabla_{\partial_7}\partial_7 &= \frac{1}{2}\Big((aa_1 - a_5)\partial_1 + (aa_2 - a_6)\partial_2 + (aa_3 + a_7)\partial_3 + (aa_4 - a_8)\partial_4\Big)\\
&\quad -\frac{1}{2}\big(a_1\partial_5 + a_2\partial_6 + a_3\partial_7 + a_4\partial_8\big),\\
\nabla_{\partial_8}\partial_8 &= \frac{1}{2}\Big((aa_1 - a_5)\partial_1 + (aa_2 - a_6)\partial_2 + (aa_3 - a_7)\partial_3 + (aa_4 + a_8)\partial_4\Big)\\
&\quad -\frac{1}{2}\big(a_1\partial_5 + a_2\partial_6 + a_3\partial_7 + a_4\partial_8\big),\\
\nabla_{\partial_5}\partial_6 &= \frac{1}{2}a_6\partial_1 + \frac{1}{2}a_5\partial_2, \qquad \nabla_{\partial_5}\partial_7 = \frac{1}{2}a_7\partial_1 + \frac{1}{2}a_5\partial_3,\\
\nabla_{\partial_5}\partial_8 &= \frac{1}{2}a_8\partial_1 + \frac{1}{2}a_5\partial_2, \qquad \nabla_{\partial_6}\partial_7 = \frac{1}{2}a_7\partial_2 + \frac{1}{2}a_6\partial_3,\\
\nabla_{\partial_6}\partial_8 &= \frac{1}{2}a_8\partial_2 + \frac{1}{2}a_6\partial_4, \qquad \nabla_{\partial_7}\partial_8 = \frac{1}{2}a_8\partial_3 + \frac{1}{2}a_7\partial_4.
\end{aligned}
$$

3.1. First class of Walker metrics

Suppose that a is a function of $(u_1, \ldots, u_4)$. From the relations (3.4), after a long but straightforward calculation, the non-zero components of the $(1,3)$-curvature operator of any Walker metric (3.1) and (3.2) is given by

(3.5)

$$
R(\partial_5, \partial_6)\partial_5 = \frac{aa_1a_2}{2}\partial_1 + \frac{aa_2^2}{2}\partial_2 + \frac{aa_2a_3}{2}\partial_3 + \frac{aa_2a_4}{2}\partial_4
$$

$$
\begin{aligned}
& -\frac{a_1a_2}{2}\partial_5-\frac{a_2^2}{2}\partial_6-\frac{a_2a_3}{2}\partial_7-\frac{a_2a_4}{2}\partial_8;\\
R(\partial_5,\partial_6)\partial_6 \;=\;& -\frac{aa_1^2}{2}\partial_1-\frac{aa_2a_2}{2}\partial_2-\frac{aa_1a_3}{2}\partial_3-\frac{aa_1a_4}{2}\partial_4\\
& +\frac{a_1^2}{2}\partial_5+\frac{a_1a_2}{2}\partial_6+\frac{a_1a_3}{2}\partial_7+\frac{a_1a_4}{2}\partial_8;\\
R(\partial_5,\partial_7)\partial_5 \;=\;& \frac{aa_1a_3}{2}\partial_1+\frac{aa_2a_3}{2}\partial_2+\frac{aa_3^2}{2}\partial_3+\frac{aa_3a_4}{2}\partial_4\\
& -\frac{a_1a_3}{2}\partial_5-\frac{a_2a_3}{2}\partial_6-\frac{a_3^2}{2}\partial_7-\frac{a_3a_4}{2}\partial_8;\\
R(\partial_5,\partial_7)\partial_7 \;=\;& -\frac{aa_1^2}{2}\partial_1-\frac{aa_1a_2}{2}\partial_2-\frac{aa_1a_3}{2}\partial_3-\frac{aa_1a_4}{2}\partial_4\\
& +\frac{a_1^2}{2}\partial_5+\frac{a_1a_2}{2}\partial_6+\frac{a_1a_3}{2}\partial_7+\frac{a_1a_4}{2}\partial_8;\\
R(\partial_5,\partial_8)\partial_5 \;=\;& \frac{aa_1a_4}{2}\partial_1+\frac{aa_2a_4}{2}\partial_2+\frac{aa_3a_4}{2}\partial_3+\frac{aa_4^2}{2}\partial_4\\
& -\frac{a_1a_4}{2}\partial_5-\frac{a_2a_4}{2}\partial_6-\frac{a_3a_4}{2}\partial_7-\frac{a_4^2}{2}\partial_8;\\
R(\partial_5,\partial_8)\partial_8 \;=\;& -\frac{aa_1^2}{2}\partial_1-\frac{aa_2a_2}{2}\partial_2-\frac{aa_1a_3}{2}\partial_3-\frac{aa_1a_4}{2}\partial_4\\
& +\frac{a_1^2}{2}\partial_5+\frac{a_1a_2}{2}\partial_6+\frac{a_1a_3}{2}\partial_7+\frac{a_4a_4}{2}\partial_8;\\
R(\partial_6,\partial_7)\partial_6 \;=\;& \frac{aa_1a_3}{2}\partial_1+\frac{aa_2a_3}{2}\partial_2+\frac{aa2_3}{2}\partial_3+\frac{aa_3a_4}{2}\partial_4\\
& -\frac{a_1a_3}{2}\partial_5-\frac{a_2a_3}{2}\partial_6-\frac{a_3^2}{2}\partial_7-\frac{a_3a_4}{2}\partial_8;\\
R(\partial_6,\partial_7)\partial_7 \;=\;& -\frac{aa_1a_3}{2}\partial_1-\frac{aa_2a_3}{2}\partial_2-\frac{aa_3^2}{2}\partial_3-\frac{aa_3a_4}{2}\partial_4\\
& +\frac{a_1a_3}{2}\partial_5+\frac{a_2a_3}{2}\partial_6+\frac{a_3^2}{2}\partial_7+\frac{a_3a_4}{2}\partial_8;\\
R(\partial_6,\partial_8)\partial_6 \;=\;& \frac{aa_1a_4}{2}\partial_1+\frac{aa_2a_4}{2}\partial_2+\frac{aa_3a_4}{2}\partial_3+\frac{aa_4^2}{2}\partial_4\\
& -\frac{a_1a_4}{2}\partial_5-\frac{a_2a_4}{2}\partial_6-\frac{a_3a_4}{2}\partial_7-\frac{a_4^2}{2}\partial_8;\\
R(\partial_6,\partial_8)\partial_8 \;=\;& -\frac{aa_1a_2}{2}\partial_1-\frac{aa_2^2}{2}\partial_2-\frac{aa_2a_3}{2}\partial_3-\frac{aa_2a_4}{2}\partial_4\\
& +\frac{a_1a_2}{2}\partial_5+\frac{a_2^2}{2}\partial_6+\frac{a_2a_3}{2}\partial_7+\frac{a_2a_4}{2}\partial_8;\\
R(\partial_7,\partial_8)\partial_7 \;=\;& \frac{aa_1a_4}{2}\partial_1+\frac{aa_2a_4}{2}\partial_2+\frac{aa_3a_4}{2}\partial_3+\frac{aa_4^2}{2}\partial_4\\
& -\frac{a_1a_4}{2}\partial_5-\frac{a_2a_4}{2}\partial_6-\frac{a_3a_4}{2}\partial_7-\frac{a_4^2}{2}\partial_8;
\end{aligned}
$$

$$\begin{aligned} R(\partial_7,\partial_8)\partial_8 &= -\frac{aa_1a_3}{2}\partial_1-\frac{aa_2a_3}{2}\partial_2-\frac{aa_3^2}{2}\partial_3-\frac{aa_3a_4}{2}\partial_4 \\ &\quad +\frac{a_1a_3}{2}\partial_5+\frac{a_2a_3}{2}\partial_6+\frac{a_3^2}{2}\partial_7+\frac{a_3a_4}{2}\partial_8. \end{aligned}$$

From the relations (3.5), after a straightforward calculation, the non-zero components of the $(0,4)$-curvature tensor of any Walker metric (3.1) and (3.2) with $a=a(u_1,\dots,u_4)$ are given by

$$\begin{aligned} R_{1556} &= R_{5662}=R_{5772}=R_{5882}=R_{6771}=R_{6881}=\frac{a_1a_2}{4}, \\ R_{1557} &= R_{5663}=R_{5773}=R_{5883}=R_{1667}=R_{7881}=\frac{a_1a_3}{4}, \\ R_{1558} &= R_{5664}=R_{5774}=R_{5884}=R_{1668}=R_{1778}=\frac{a_1a_4}{4}, \\ R_{2557} &= R_{5635}=R_{6773}=R_{6883}=R_{2667}=R_{7882}=\frac{a_2a_3}{4}, \\ R_{2558} &= R_{5645}=R_{6774}=R_{6884}=R_{2668}=R_{2778}=\frac{a_2a_4}{4}, \\ R_{3558} &= R_{5745}=R_{4667}=R_{3668}=R_{3778}=R_{7884}=\frac{a_3a_4}{4}, \\ R_{1665} &= R_{1775}=R_{1885}=\frac{a_1^2}{4}, \quad R_{2556}=R_{2776}=R_{2886}=\frac{a_2^2}{4}, \\ R_{3557} &= R_{3667}=R_{3887}=\frac{a_3^2}{4}, \quad R_{4558}=R_{4668}=R_{4778}=\frac{a_4^2}{4}, \end{aligned} \tag{3.6}$$

Next, let $\rho(X,Y)=\operatorname{trace}\{Z\longrightarrow R(X,Z)Y\}$ be the Ricci tensor. Then from (3.6) we have

$$\begin{aligned} \rho_{55} &= \frac{1}{2}(a_2^2+a_3^2+a_4^2), \quad \rho_{66}=\frac{1}{2}(a_1^2+a_3^2+a_4^2) \\ \rho_{77} &= \frac{1}{2}(a_1^2+a_2^2+a_4^2), \quad \rho_{88}=\frac{1}{2}(a_1^2+a_2^2+a_3^2) \\ \rho_{56} &= -\frac{a_1a_2}{2}, \quad \rho_{57}=-\frac{a_1a_3}{2}, \quad \rho_{58}=-\frac{a_1a_4}{2} \\ \rho_{67} &= -\frac{a_2a_3}{2}, \quad \rho_{68}=-\frac{a_2a_4}{2}, \quad \rho_{78}=-\frac{a_3a_4}{2}. \end{aligned} \tag{3.7}$$

From (3.7), the scalar curvature $Sc=\sum_1^8 g^{ij}\rho_{ij}$ of the Walker metric is zero. We have the following result.

Theorem 3.1. *A Walker metric given by* (3.1) *and* (3.2) *is not Einstein if the function a depends only on* $(u_1,\dots,u_4)$.

Proof. The Einstein equations defined by $G_{ij}=\rho_{ij}-\frac{Sc}{8}g_{ij}$ for the Walker metric given by (3.1) and (3.2) with $a=a(u_1,\dots,u_4)$ are as follows:

$$\begin{aligned} G_{56} &= -\frac{a_1a_2}{2}=0, \quad G_{57}=-\frac{a_1a_3}{2}=0, \quad G_{58}=-\frac{a_1a_4}{2}=0 \\ G_{67} &= -\frac{a_2a_3}{2}=0, \quad G_{68}=-\frac{a_2a_4}{2}=0, \quad G_{78}=-\frac{a_3a_4}{2}, \end{aligned}$$

and

$$\begin{aligned} G_{55} &= \frac{1}{2}(a_2^2+a_3^2+a_4^2)=0, \quad G_{66}=\frac{1}{2}(a_1^2+a_3^2+a_4^2)=0 \\ G_{77} &= \frac{1}{2}(a_1^2+a_2^2+a_4^2)=0, \quad G_{88}=\frac{1}{2}(a_1^2+a_2^2+a_3^2)=0. \end{aligned}$$

This completes the proof. □

Let W denote the Weyl conformal curvature tensor given by

$$\begin{aligned} W(X,Y,Z,T): &= R(X,Y,Z,T) \\ &+ \frac{Sc}{(n-1)(n-2)}\Big\{g(Y,Z)g(X,T)-g(X,Z)g(Y,T)\Big\} \\ &+ \frac{1}{n-2}\Big\{\rho(Y,Z)g(X,T)-\rho(X,Z)g(Y,T) \\ &- \rho(Y,T)g(X,Z)+\rho(X,T)g(Y,Z)\Big\}. \end{aligned}$$

A pseudo-Riemannian manifold is *locally conformally flat* if and only if its Weyl tensor vanishes. The non-zero components of Weyl conformal tensor of a Walker metric defined by (3.1) and (3.2) with $a=a(u_1,\dots,u_4)$ are given by

$$\begin{aligned} (3.8)\quad W_{1556} &= \frac{a_1a_2}{3}, \quad W_{1557}=\frac{a_1a_3}{3}, \quad W_{1558}=\frac{a_1a_4}{3}, \\ W_{1665} &= \frac{1}{12}(4a_1^2+a_3^2+a_4^2), \quad W_{1667}=\frac{a_1a_3}{4}, \quad W_{1668}=\frac{a_1a_4}{4}, \\ W_{1775} &= \frac{1}{12}(4a_1^2+a_2^2+a_4^2), \quad W_{1776}=\frac{a_1a_2}{4}, \quad W_{1778}=\frac{a_1a_4}{4}, \\ W_{1885} &= \frac{1}{12}(4a_1^2+a_2^2+a_3^2), \quad W_{1886}=\frac{a_1a_2}{4}, \quad W_{1887}=\frac{a_1a_3}{4}, \\ W_{2556} &= \frac{1}{12}(4a_2^2+a_3^2+a_4^2), \quad W_{2557}=\frac{a_2a_3}{4}, \quad W_{2558}=\frac{a_2a_4}{4}, \\ W_{2665} &= \frac{a_1a_2}{3}, \quad W_{2667}=\frac{a_2a_3}{3}, \quad W_{2668}=\frac{a_2a_4}{3}, \\ W_{2775} &= \frac{a_1a_2}{4}, \quad W_{2776}=\frac{1}{12}(a_1^2+4a_2^2+a_4^2), \quad W_{2778}=\frac{a_2a_4}{4}, \\ W_{2885} &= \frac{a_1a_2}{4}, \quad W_{2886}=\frac{1}{12}(a_1^2+4a_2^2+a_3^2), \quad W_{2887}=\frac{a_2a_3}{4}, \\ W_{3556} &= \frac{a_2a_3}{4}, \quad W_{3557}=\frac{1}{12}(a_2^2+4a_3^2+a_4^2), \quad W_{3558}=\frac{a_3a_4}{4}, \\ W_{3665} &= \frac{a_1a_3}{4}, \quad W_{3667}=\frac{1}{12}(a_1^2+4a_3^2+a_4^2), \quad W_{3668}=\frac{a_3a_4}{4}, \\ W_{3775} &= \frac{a_1a_3}{3}, \quad W_{3776}=\frac{a_2a_3}{3}, \quad W_{3778}=\frac{a_3a_4}{3}, \\ W_{3885} &= \frac{a_1a_3}{4}, \quad W_{3886}=\frac{a_2a_3}{4}, \quad W_{3887}=\frac{1}{12}(a_1^2+a_2^2+4a_3^2), \\ W_{4556} &= \frac{a_2a_4}{4}, \quad W_{4557}=\frac{a_3a_4}{4}, \quad W_{4558}=\frac{1}{12}(a_2^2+a_3^2+4a_4^2), \end{aligned}$$

$$\begin{aligned}
W_{4665} &= \frac{a_1a_4}{4}, \quad W_{4667} = \frac{a_3a_4}{4}, \quad W_{4668} = \frac{1}{12}(a_1^2+a_3^2+4a_4^2),\\
W_{4775} &= \frac{a_1a_4}{4}, \quad W_{4776} = \frac{a_2a_4}{4}, \quad W_{4778} = \frac{1}{12}(a_1^2+a_2^2+4a_4^2),\\
W_{4885} &= \frac{a_1a_4}{3}, \quad W_{4886} = \frac{a_2a_4}{3}, \quad W_{4887} = \frac{a_3a_4}{3}.
\end{aligned}$$

Now it is possible to obtain the form of a locally conformally flat Walker metric defined by (3.1) and (3.2) with $a = a(u_1, \ldots, u_4)$.

Theorem 3.2. *A Walker metric given by* (3.1) *and* (3.2) *with* $a = a(u_1, \ldots, u_4)$ *is locally conformally flat if the function* a *is constant.*

Proof. From (3.8) after a straightforward calculation. □

3.2. Strictly Walker metrics

One says that (M, g) is a *strict Walker manifold* if a is a function of the $(u_5, \ldots, u_8)$. From the relations (3.4), after a long but straightforward calculation, the non-zero components of the (1, 3)-curvature operator of any Walker metric (3.1) and (3.2) with $a = a(u_5, \ldots, u_8)$ are given by

(3.9)

$$\begin{aligned}
R(\partial_5,\partial_6)\partial_5 &= \frac{1}{2}(a_{55}+a_{66})\partial_2 + \frac{1}{2}a_{67}\partial_3 + \frac{1}{2}a_{68}\partial_4,\\
R(\partial_5,\partial_6)\partial_6 &= -\frac{1}{2}(a_{55}+a_{66})\partial_1 - \frac{1}{2}a_{57}\partial_3 - \frac{1}{2}a_{58}\partial_4,\\
R(\partial_5,\partial_6)\partial_7 &= -\frac{1}{2}a_{67}\partial_1 + \frac{1}{2}a_{57}\partial_2, \quad R(\partial_5,\partial_6)\partial_8 = -\frac{1}{2}a_{68}\partial_1 + \frac{1}{2}a_{58}\partial_2,\\
R(\partial_5,\partial_7)\partial_5 &= \frac{1}{2}a_{76}\partial_2 + \frac{1}{2}(a_{55}+a_{77})\partial_3 + \frac{1}{2}a_{78}\partial_4,\\
R(\partial_5,\partial_7)\partial_7 &= -\frac{1}{2}(a_{55}+a_{77})\partial_1 - \frac{1}{2}a_{56}\partial_2 - \frac{1}{2}a_{58}\partial_4,\\
R(\partial_5,\partial_7)\partial_6 &= -\frac{1}{2}a_{76}\partial_1 + \frac{1}{2}a_{56}\partial_3, \quad R(\partial_5,\partial_7)\partial_8 = -\frac{1}{2}a_{78}\partial_1 + \frac{1}{2}a_{58}\partial_3,\\
R(\partial_5,\partial_8)\partial_5 &= \frac{1}{2}a_{86}\partial_2 + \frac{1}{2}a_{87}\partial_3 + \frac{1}{2}(a_{55}+a_{88})\partial_4,\\
R(\partial_5,\partial_8)\partial_6 &= -\frac{1}{2}a_{86}\partial_1 + \frac{1}{2}a_{56}\partial_4, \quad R(\partial_5,\partial_8)\partial_7 = -\frac{1}{2}a_{87}\partial_1 + \frac{1}{2}a_{57}\partial_4,\\
R(\partial_5,\partial_8)\partial_8 &= -\frac{1}{2}(a_{55}+a_{88})\partial_1 - \frac{1}{2}a_{56}\partial_2 - \frac{1}{2}a_{57}\partial_3,\\
R(\partial_6,\partial_7)\partial_6 &= \frac{1}{2}a_{75}\partial_1 + \frac{1}{2}(a_{66}+a_{77})\partial_3 + \frac{1}{2}a_{78}\partial_4,\\
R(\partial_6,\partial_7)\partial_5 &= -\frac{1}{2}a_{75}\partial_2 + \frac{1}{2}a_{65}\partial_3, \quad R(\partial_6,\partial_7)\partial_8 = -\frac{1}{2}a_{78}\partial_2 + \frac{1}{2}a_{68}\partial_3,\\
R(\partial_6,\partial_7)\partial_7 &= -\frac{1}{2}a_{65}\partial_1 - \frac{1}{2}(a_{66}+a_{77})\partial_2 - \frac{1}{2}a_{68}\partial_4,\\
R(\partial_7,\partial_8)\partial_5 &= -\frac{1}{2}a_{85}\partial_3 + \frac{1}{2}a_{75}\partial_4, \quad R(\partial_7,\partial_8)\partial_6 = -\frac{1}{2}a_{86}\partial_3 + \frac{1}{2}a_{76}\partial_4,
\end{aligned}$$

$$\begin{aligned} R(\partial_7,\partial_8)\partial_7 &= \frac{1}{2}a_{85}\partial_1 + \frac{1}{2}a_{85}\partial_2 + \frac{1}{2}(a_{77}+a_{88})\partial_4, \\ R(\partial_7,\partial_8)\partial_8 &= -\frac{1}{2}a_{75}\partial_1 - \frac{1}{2}a_{76}\partial_2 - \frac{1}{2}(a_{77}+a_{88})\partial_3. \end{aligned}$$

From the relations (3.9), after a straightforward calculation, the non-zero components of the $(0,4)$-curvature tensor of any Walker metric (3.1) and (3.2) with $a = a(u_5,\dots,u_8)$ are obtained as

$$\begin{aligned} R_{5657} &= R_{6878} = \frac{1}{2}a_{67}, \quad R_{5658} = R_{6787} = \frac{1}{2}a_{68}, \\ R_{5676} &= R_{5878} = \frac{1}{2}a_{57}, \quad R_{5686} = R_{5787} = \frac{1}{2}a_{58}, \\ R_{5758} &= R_{6768} = \frac{1}{2}a_{78}, \quad R_{5767} = R_{5868} = \frac{1}{2}a_{56}, \\ R_{5656} &= \frac{1}{2}(a_{55}+a_{66}), \quad R_{5757} = \frac{1}{2}(a_{55}+a_{77}), \\ R_{5858} &= \frac{1}{2}(a_{55}+a_{88}), \quad R_{6767} = \frac{1}{2}(a_{66}+a_{77}), \\ R_{6868} &= \frac{1}{2}(a_{66}+a_{88}), \\ R_{7878} &= \frac{1}{2}(a_{77}+a_{88}). \end{aligned} \tag{3.10}$$

We have the following:

Theorem 3.3. *Let (M,g) be as in* (3.1) *and* (3.2) *with $a = a(u_5,\dots,u_8)$. Then the following holds:*

1. *(M,g) is Ricci flat.*

2. *(M,g) is locally conformally flat if and only if (M,g) is flat. This means the function $a = a(u_5,\dots,u_8)$ is a constant.*

Proof. From the formula $\rho_{kl} = \sum_{i,j=1}^{8} g^{ij}R_{kijl}$ all the components of Ricci tensor are zero. □

4. Almost Kähler Walker 8-manifolds

An almost Hermitian structure on a manifold M consists of a non-degenerate 2-form Ω, an almost complex structure J and a metric g satisfying the compatibility condition $\Omega(X,Y) = g(JX,Y)$. If the 2-form Ω is closed (i.e., it is a symplectic form) the structure is said to be almost Kähler and (g,J) is said to be Kähler if, in addition, the almost complex structure J is integrable (i.e., it is defined by a complex coordinate atlas on M). It is worth emphasizing that each two of the objects (g,J,Ω) determine the third one. However, whenever the starting point is a symplectic structure Ω, there are many different pairs (g,J) of almost Hermitian structures sharing the same Kähler form Ω.

Given a Walker 8-manifold M with a metric g, we can construct various g-orthogonal almost complex structures J on M so that (M, g, J) is almost Hermitian. We can define an almost complex structure as follows:

$$\begin{aligned} J\partial_1 &= \partial_2, \quad J\partial_3 = \partial_4, \quad J\partial_2 = -\partial_1, \quad J\partial_4 = -\partial_3, \\ J\partial_5 &= \partial_6, \quad J\partial_7 = \partial_8, \quad J\partial_6 = -\partial_5, \quad J\partial_8 = -\partial_7. \end{aligned} \tag{4.1}$$

From the actions of J on the $\partial_i, i = 1, \ldots, 8$, if we write $J\partial_i = \sum_{j=1}^{8} J_i^j \partial_j$, then we can have the non-zero components J_i^j of J as follows:

$$J_1^2 = -J_2^1 = J_3^4 = -J_4^3 = J_5^6 = -J_6^5 = J_7^8 = -J_8^7 = 1. \tag{4.2}$$

Next, we shall study this almost Hermitian structure (g, J) on $\mathbb{R}^8$, with g as in (3.1), (3.2) and J as in (4.1).

Associated with the almost Hermitian structure (g, J) is the Kähler form Ω, defined by $\Omega(X, Y) = g(JX, Y)$ for any vector fields X, Y with coordinate expression given by

$$\begin{aligned} \Omega &= \sum_{i<j} \Omega(\partial_i, \partial_j) du^i \wedge du^j \\ &= du^1 \wedge du^2 + du^1 \wedge du^6 - du^2 \wedge du^5 + du^3 \wedge du^4 \\ &+ du^3 \wedge du^8 - du^4 \wedge du^7 + a du^5 \wedge du^6 + a du^7 \wedge du^8. \end{aligned} \tag{4.3}$$

We can compute the differential of Ω as follows:

$$\begin{aligned} d\Omega = {} & a_1\big(du^1 \wedge du^5 \wedge du^6 + du^1 \wedge du^7 \wedge du^8\big) \\ & + a_2\big(du^2 \wedge du^5 \wedge du^6 + du^2 \wedge du^7 \wedge du^8\big) \\ & + a_3\big(du^3 \wedge du^5 \wedge du^6 + du^3 \wedge du^7 \wedge du^8\big) \\ & + a_4\big(du^4 \wedge du^5 \wedge du^6 + du^4 \wedge du^7 \wedge du^8\big) \\ & + a_5 du^5 \wedge du^7 \wedge du^8 + a_6 du^6 \wedge du^7 \wedge du^8 \\ & + a_7 du^7 \wedge du^5 \wedge du^6 + a_8 du^8 \wedge du^5 \wedge du^6, \end{aligned} \tag{4.4}$$

where $a_i = \partial a/\partial_i, i = 1, \ldots, 8$. From this expression, we have the following

Proposition 4.1. *The 2-form Ω is symplectic if and only if the function a is a constant.*

The almost complex structure J is integrable if and only if the torsion of J (Nijenhuis tensor) vanishes, i.e., the components

$$N_{jk}^i = 2\sum_{h=1}^{8} \left(J_j^h \frac{\partial J_k^i}{\partial u^h} - J_k^h \frac{\partial J_j^i}{\partial u^h} - J_h^i \frac{\partial J_k^h}{\partial u^j} + J_h^i \frac{\partial J_j^h}{\partial u^h} \right) \tag{4.5}$$

all vanish (cf. [11]) with J_i^j as in (4.2). By explicit calculation, all components of the Nijenhuis tensor vanish. Recall that, the almost Hermitian structure (g, J) is Kähler if the 2-form is a symplectic form and J is integrable. Thus we have:

Theorem 4.2. *The almost Hermitian Walker* 8*-manifold* (M, g, J)*, with* g *as in* (3.1)*,* (3.2) *and* J *as in* (4.1)*, is Kähler if and only if the function* a *is a constant.*

A long standing problem in almost Hermitian geometry is that of relating the properties of the structure (g, J, Ω) to the curvature of (M, g). For example the Goldberg conjecture [7], which claims that if a compact almost Kähler manifold is Einstein, then it is Kähler. This conjecture was proved by K. Sekigawa [12], in the case of non-negative scalar curvature, but it remains open in the negative case. Although the Goldberg conjecture is of global nature, it is known that some additional curvature conditions suffice to show the integrability of the almost complex structure at the local level.

Remark 4.3. Y. Matsushita [9] considered a restricted class of Walker 4-manifolds, by imposing a restriction on the general expression of the canonical form of the metric of a Walker 4-manifold. On this restricted class of Walker 4-manifolds Matsushita constructed an almost complex structure and an opposite almost complex structure which commute and considers the associated Kähler forms. Then he gave conditions for the Walker 4-manifold to admit a symplectic structure, the almost complex structure to be integrable and the metric to be Einstein. That is, Matsushita shows that the class of Walker 4-manifolds studied contains examples of indefinite Kähler-Einstein 4-manifolds, examples of indefinite Hermitian 4-manifolds and examples of indefinite almost Kähler 4-manifolds. Moreover, he also gave conditions for the opposite Kähler form to be a symplectic form.

Matsushita [10] studied the Walker 4-manifold as an almost Hermitian 4-manifold. The almost Kähler condition, the Hermitian condition and the Kähler condition for the almost Hermitian structure (g, J) are explicitly given in terms of three functions a, b and c characterizing the metric g. If the almost Hermitian structure (g, J) is Kähler, then these functions a, b and c are all harmonic with respect to two coordinates. From this fact, for any given harmonic function of two variables, an indefinite Kähler metric can be constructed on a Walker 4-manifold, thereby giving a family of indefinite Kähler 4-manifolds. It should be noted that, in this family, a specific Kähler metric thus constructed is nothing but Petean's non-flat indefinite Kähler-Einstein metric on a complex torus. The paper [10] also includes a counterexample of indefinite and non-compact type, constructed by Haze on a Walker 4-manifold, to the Goldberg conjecture [7]. The Walker 4-manifold with an opposite almost complex structure is also analyzed as an opposite almost Hermitian 4-manifold.

In [5], it is proved that any proper almost Hermitian structure on a Walker four-manifold is isotropic Kähler. As examples, isotropic Kähler, almost Kähler and Hermitian structures can be defined on tori. Moreover, for almost Kähler Walker four-manifolds which are self-dual, -Einstein or Einstein, local descriptions are given. As a consequence of such descriptions, it is shown that any proper almost Kähler Einstein structure is self-dual, Ricci flat and $*$-Ricci flat. This is used to supply examples of flat indefinite non-Kähler and almost Kähler structures.

The authors in [4] provided a large family of non-Kähler isotropic Kähler Hermitian structures having interesting curvature properties. They considered Walker metrics on 4-manifolds together with the proper almost complex structure and obtained a local description of those metrics which are Hermitian or locally conformally Kähler and self-dual, -Einstein or Einstein. They also constructed examples of indefinite Einstein strictly almost Hermitian structures showing that the integrability result given by Kirchberg in [8] does not hold for metrics of signature $(2, 2)$.

In [6], the authors studied a particular almost complex structure J. For this, they explicitly solved the PDEs for the fundamental 2-form to be closed (the almost Kähler condition), and they gave the integrability condition of J, which looks similar to the almost Kähler conditions. They also obtained Walker metrics which can be Hermitian with respect to such J.

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