Efficient implementation of Some Block Extended Trapezoidal rule of first kind (ETRs) class of method Applied to First Order Initial Value Problems of Differential Equations

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ABSTRACT

The numerical solution of first order ordinary differential equations is here approached by approximating a class of $k=3$, $k=5$ and $k=7$ step block extended trapezoidal rule of first kind respectively. The stability properties of the method discussed. Some numerical test, reported to show efficiency and accuracy of our newly developed method.

Keywords: First order ODE, Extended Trapezoidal Rule, A-Stability, Interpolation and Collocation, Continuous Formulation.

1. Introduction

This paper seeks the construction of a class of extended trapezoidal rule of first kind for direct integration of initial value problem of the form

$$y'(x) = f(x, y(x)),\ y(x_0) = y_0 \tag{1.1}$$

on the finite interval $I=[x_0, x_N]$ where, $y: I \to R^m$ and $f: I \times R^m \to R^m$ is continuous and differentiable.

In recent years the problems of deriving more advanced and efficient methods for (1.1) has received a great deal of attention, and as a result a wide variety of approaches have been proposed. A potential good numerical method for the solution of (1.1) must have good accuracy and some reasonably wide region of absolute stability [Dahlquist, 1963; Enright, 1974; Lambert, 1972; Boroni et al., 2009].

To solve (1.1) numerically, we use the continuous formulation

$$y(x_n + (t+1)h) = \alpha_v(t) y_{n+v} + \sum_{j=0}^{k} \beta_{i,j}(t) f_{n+j} \tag{1.2}$$ to

compute the approximate solution y_{n+j} to $y(x_{n+j})$ at the end of a step $[x_n, x_{n+1}]$ of length $h = x_{n+1} - x_n$ for each $n = 0,1,2,\ldots,$ and $f_{n+j} = f(x_{n+j}, y_{n+j})$. The continuous coefficients $\{\alpha_v(t), \beta_{1,j}(t), j = 0(1)k\}$ in x are presumed to be real and satisfying the normalization condition $\alpha_v(t) = 1$ and the variable t in $t = x - x_{n+1}/h$.

Consider equation (2.5), writing this in block form we have

$$A^{(0)} Y_m = A^{(1)} y_n + hdf(y_n) + hbF(Y_m) \tag{1.3}$$

where,

$$Y_m = [y_{i+1}]^T,\ y_n = [y_{i-2}]^T,\ F(Y_m) = [f_{i+1}]^T,\ f(y_n) = [f_{i-2}]^T$$

and

$$i = n(n+1)n + 2$$

The linear operator $L\{y(x);h\}$ associated with the block (1.3) can be defined as:

$$L\{y(x);h\} = A^{(0)} Y_m - A^{(1)} y_n - hdf(y_n) - hbF(Y_m) \tag{1.4}$$

where $y(x_n)$ is any sufficiently differentiable vector valued function. By Taylor series expansion, we have that

$$L\{y(x);h\}=c_{p+1}h^{(p+1)}y^{(p+1)}(x_n)+0(h^{p+2}),x\in[x_n,x_{n+1}] \quad (1.5)$$

where, c is regarded as the error constants, p the order of (1.2).

2. Construction of the ETRs Method

The solution of the initial value problems (1.1) is approximated by the polynomial

$$y(x)=\sum_{j=0}^{k+1}a_jx^j \quad (2.1)$$

where, $\{a_j\}_{j=0}^{k+1}$ are real parameter constants to be determined. From (1.5), we obtained

$$y'(x)=f(x,y)=\sum_{j=1}^{k+1}ja_jx^{j-1} \quad (2.2)$$

Interpolating (2.1) at $x=x_{n+v-1}$ and collocating (2.2) at $x=x_{n+j}$, $j=0(1)k$, we obtain the linear system of equations

$$\begin{bmatrix} 1 & x_{n+v-1} & x_{n+v-1}^2 & x_{n+v-1}^3 & . & . & . & x_{n+v-1}^{k+1} \\ 0 & 1 & 2x_n & 3x_n^2 & . & . & . & (k+1)x_n^k \\ 0 & & 2x_{n+1} & 3x_{n+1}^2 & . & . & . & (k+1)x_{n+1}^k \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ 0 & 1 & 2x_{n+k} & 3x_{n+k}^2 & . & . & . & (k+1)x_{n+k}^k \end{bmatrix} \quad (2.3)$$

Solving equation (2.3) for a_j, s and substituting the resulting values into (2.1) with $x=x_{n+1}+th$, setting $x=x_{n+t+1}$ on the left hand-side of (2.1) yield the continuous formulation for $k=3(5)7$ and $t=k-1$.

a. For $v=1,t=0$; It is exactly the well known implicit

$$2y_{i+1}-2y_i=hf(x_{i+1},y_{i+1})+hf(x_i,y_i) \quad (2.4)$$

Trapezoidal rule.

a.

b. For $v=t=2$; The implicit and explicit three step ETRs equations are obtained

$$\left.\begin{aligned} 3y_{i+3}-3y_{i+1}&=hf(x_{i+3},y_{i+3})+4hf(x_{i+2},y_{i+2})+hf(x_{i+1},y_{i+1}) \\ 24y_{i+2}-24y_{i+1}&=-hf(x_{i+3},y_{i+3})+13hf(x_{i+2},y_{i+2})+13hf(x_{i+1},y_{i+1})-hf(x_i,y_i) \\ 24y_{i+1}-24y_i&=hf(x_{i+3},y_{i+3})-5hf(x_{i+2},y_{i+2})+19hf(x_{i+1},y_{i+1})+9hf(x_i,y_i) \end{aligned}\right\} \quad (2.5)$$

c. For $v=3, t=4$; We obtained the following implicit and explicit five step ETRs equations

$$90y_{i+2}-90y_{i}=hf(x_{i+5},y_{i+5})-6hf(x_{i+4},y_{i+4})+14hf(x_{i+3},y_{i+3})$$
$$+14hf(x_{i+2},y_{i+2})+129hf(x_{i+1},y_{i+1})+28hf(x_{i},y_{i})$$

$$1440y_{i+2}-1440y_{i+1}=-11hf(x_{i+5},y_{i+5})+77hf(x_{i+4},y_{i+4})-258hf(x_{i+3},y_{i+3})$$
$$+1022hf(x_{i+2},y_{i+2})+637hf(x_{i+1},y_{i+1})-27hf(x_{i},y_{i})$$

$$1440y_{i+3}-1440y_{i+2}=11hf(x_{i+5},y_{i+5})-93hf(x_{i+4},y_{i+4})+802hf(x_{i+3},y_{i+3})$$
$$+802hf(x_{i+2},y_{i+2})-93hf(x_{i+1},y_{i+1})+11hf(x_{i},y_{i})$$

$$90y_{i+4}-90y_{i+2}=-hf(x_{i+5},y_{i+5})+34hf(x_{i+4},y_{i+4})+11hf(x_{i+3},y_{i+3})$$
$$+34hf(x_{i+2},y_{i+2})-hf(x_{i+1},y_{i+1})$$

$$160y_{i+3}-160y_{i+2}=51hf(x_{i+5},y_{i+5})+219hf(x_{i+4},y_{i+4})+114hf(x_{i+3},y_{i+3})$$
$$+114hf(x_{i+2},y_{i+2})-21hf(x_{i+1},y_{i+1})+3hf(x_{i},y_{i}) \quad (2.6)$$

d. For $v=4, t=6$; The following implicit and explicit seven step ETRs equations are obtained

$$\left.\begin{aligned}
&4480y_{i+3}-4480y_{i}=45hf(x_{i+7},y_{i+7})-373hf(x_{i+6},y_{i+6})+1377hf(x_{i+5},y_{i+5})\\
&-3033hf(x_{i+4},y_{i+4})+5927hf(x_{i+3},y_{i+3})+1377hf(x_{i+2},y_{i+2})\\
&+6795hf(x_{i+1},y_{i+1})+1325hf(x_{i},y_{i})\\
&3780y_{i+3}-3780y_{i+1}=-5hf(x_{i+7},y_{i+7})+40hf(x_{i+6},y_{i+6})-135hf(x_{i+5},y_{i+5})\\
&+208hf(x_{i+4},y_{i+4})+1153hf(x_{i+3},y_{i+3})+4968hf(x_{i+2},y_{i+2})\\
&+1363hf(x_{i+1},y_{i+1})-32hf(x_{i},y_{i})\\
&120960y_{i+3}-120960y_{i+2}=191hf(x_{i+7},y_{i+7})-1719hf(x_{i+6},y_{i+6})\\
&+7227hf(x_{i+5},y_{i+5})-20227hf(x_{i+4},y_{i+4})+81693hf(x_{i+3},y_{i+3})\\
&+57627hf(x_{i+2},y_{i+2})-4183hf(x_{i+1},y_{i+1})+351hf(x_{i},y_{i})\\
&120960y_{i+4}-120960y_{i+3}=-191hf(x_{i+7},y_{i+7})+1879hf(x_{i+6},y_{i+6})\\
&-9531hf(x_{i+5},y_{i+5})+68323hf(x_{i+4},y_{i+4})+68323hf(x_{i+3},y_{i+3})\\
&-9531hf(x_{i+2},y_{i+2})+1879hf(x_{i+1},y_{i+1})-191hf(x_{i},y_{i})\\
&3780y_{i+5}-3780y_{i+3}=5hf(x_{i+7},y_{i+7})-72hf(x_{i+6},y_{i+6})+1503hf(x_{i+5},y_{i+5})\\
&+4688hf(x_{i+4},y_{i+4})+1503hf(x_{i+3},y_{i+3})-72hf(x_{i+2},y_{i+2})\\
&+5hf(x_{i+1},y_{i+1})\\
&4480y_{i+6}-4480y_{i+3}=-45hf(x_{i+7},y_{i+7})+1685hf(x_{i+6},y_{i+6})+5535hf(x_{i+5},y_{i+5})\\
&+3897hf(x_{i+4},y_{i+4})+2777hf(x_{i+3},y_{i+3})-5137hf(x_{i+2},y_{i+2})\\
&+117hf(x_{i+1},y_{i+1})-13hf(x_{i},y_{i})\\
&945y_{i+7}-945y_{i+3}=278hf(x_{i+7},y_{i+7})+1448hf(x_{i+6},y_{i+6})+216hf(x_{i+5},y_{i+5})\\
&+1784hf(x_{i+4},y_{i+4})-106hf(x_{i+3},y_{i+3})+216hf(x_{i+2},y_{i+2})\\
&-64hf(x_{i+1},y_{i+1})+8hf(x_{i},y_{i})
\end{aligned}\right\}\qquad(2.7)$$

The class of methods we have developed shall each be implemented in block form to generate approximate solutions $y_i, i=0,1,2,...$ simultaneously without the need for starters, making its computation competitive. However, the only block ETRs method that was shown to be A-stable is the three step block ETRs (2.5) when $v=t=2$. For the A-stability of this method see figure 7.

3. Analysis of Basic Properties of the Methods

A. Order of the block ETRs for (2.5)

Applying (1.5) on (2.5), we obtained

$$L\{y(x);h\}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}y_{n+1}\\y_{n+2}\\y_{n+3}\end{bmatrix}-\begin{bmatrix}1&0&0\\0&1&0\\0&1&0\end{bmatrix}\begin{bmatrix}y_{n}\\y_{n+1}\\y_{n+2}\end{bmatrix}$$
$$-\begin{bmatrix}-\frac{9}{24}&-\frac{19}{24}&\frac{5}{24}&-\frac{1}{24}\\-\frac{1}{24}&\frac{13}{24}&\frac{13}{24}&-\frac{1}{24}\\0&\frac{1}{3}&\frac{4}{3}&\frac{1}{3}\end{bmatrix}\begin{bmatrix}f_n\\f_{n+1}\\f_{n+2}\\f_{n+3}\end{bmatrix} \quad (3.1)$$

Expanding (3.1) in Taylor Series gives

$$\left.\begin{array}{l}\sum_{j=0}^{\infty}\frac{(h)^j}{j!}y_n^j-y_n+\frac{9h}{24}y_n'-\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left\{-\frac{19}{24}(1)^j+\frac{5}{24}(2)^j-\frac{1}{24}(3)^j\right\}\\ \sum_{j=0}^{\infty}\frac{(2h)^j}{j!}y_n^j-y_n^{j+1}+\frac{h}{24}y_n'-\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left\{\frac{13}{24}(1)^j+\frac{13}{24}(2)^j-\frac{1}{24}(3)^j\right\}\\ 1\sum_{j=0}^{\infty}\frac{(h)^j}{j!}y_n^j-y_n^{j+1}-\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left\{\frac{1}{3}(1)^j+\frac{4}{3}(2)^j-\frac{1}{3}(3)^j\right\}\end{array}\right\}=\begin{bmatrix}0\\0\\0\end{bmatrix} \quad (3.2)$$

Following (3.2), we obtained the order and error constants of (2.5) as $p=[4 \quad 4 \quad 4]^T$ and $c_{p+1}=(-1.11(02) \quad 1.53(02) \quad -2.64(02))^T$ respectively.

B. Order of the block ETRs for (2.6)

Applying (1.5) on (2.6), we obtained

$$L\{y(x);h\}=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}\begin{bmatrix}y_{n+1}\\y_{n+2}\\y_{n+3}\\y_{n+4}\\y_{n+5}\end{bmatrix}-\begin{bmatrix}0&0&1&0&0\\1&0&0&0&0\\0&0&1&0&0\\0&0&1&0&0\\0&0&1&0&0\end{bmatrix}\begin{bmatrix}y_{n}\\y_{n+1}\\y_{n+2}\\y_{n+3}\\y_{n+4}\end{bmatrix}$$

$$-\begin{bmatrix}\frac{27}{1440}&-\frac{637}{1440}&-\frac{1002}{1440}&\frac{258}{1440}&-\frac{77}{1440}&\frac{11}{1440}\\ \frac{28}{90}&\frac{129}{90}&\frac{14}{90}&\frac{14}{90}&-\frac{6}{90}&\frac{1}{90}\\ \frac{11}{1440}&-\frac{93}{1440}&\frac{802}{1440}&\frac{802}{1440}&-\frac{93}{1440}&\frac{11}{1440}\\ 0&-\frac{1}{90}&\frac{34}{90}&\frac{114}{90}&\frac{34}{90}&-\frac{1}{90}\\ \frac{3}{160}&-\frac{21}{160}&\frac{114}{160}&\frac{114}{160}&\frac{219}{160}&\frac{51}{160}\end{bmatrix}\begin{bmatrix}f_n\\f_{n+1}\\f_{n+2}\\f_{n+3}\\f_{n+4}\\f_{n+5}\end{bmatrix} \qquad (3.3)$$

Expanding (3.3) in Taylor Series gives

$$\sum_{j=0}^{\infty}\frac{(h)^j}{j!}y_n^j-y_n^{j+2}-\frac{27h}{1440}y_n'-\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left\{\begin{matrix}-\frac{637}{1440}(1)^j-\frac{1022}{1440}(2)^j+\frac{258}{1440}(3)^j\\-\frac{77}{1440}(4)^j+\frac{11}{1440}(5)^j\end{matrix}\right\}$$

$$\sum_{j=0}^{\infty}\frac{(2h)^j}{j!}y_n^j-y_n-\frac{28h}{90}y_n'-\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left\{\begin{matrix}\frac{129}{90}(1)^j+\frac{14}{90}(2)^j+\frac{14}{90}(3)^j\\-\frac{6}{90}(4)^j+\frac{1}{90}(5)^j\end{matrix}\right\}$$

$$\sum_{j=0}^{\infty}\frac{(3h)^j}{j!}y_n^j-y_n^{j+2}-\frac{11h}{1440}y_n'-\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left\{\begin{matrix}-\frac{93}{1440}(1)^j+\frac{802}{1440}(2)^j+\frac{802}{1440}(3)^j\\-\frac{93}{1440}(4)^j+\frac{11}{1440}(5)^j\end{matrix}\right\}$$

$$\sum_{j=0}^{\infty}\frac{(4h)^j}{j!}y_n^j-y_n^{j+2}-\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left\{\begin{matrix}-\frac{1}{90}(1)^j+\frac{34}{90}(2)^j+\frac{114}{90}(3)^j\\+\frac{34}{90}(4)^j-\frac{1}{90}(5)^j\end{matrix}\right\}$$

$$\sum_{j=0}^{\infty}\frac{(5h)^j}{j!}y_n^j - y_n^{j+2} - \frac{3}{160}y_n' - \sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\begin{Bmatrix} -\frac{21}{160}(1)^j+\frac{114}{160}(2)^j+\frac{114}{160}(3)^j \\ +\frac{219}{160}(4)^j+\frac{51}{160}(5)^j \end{Bmatrix}$$

$$=\begin{bmatrix}0 & 0 & 0 & 0 & 0\end{bmatrix}^T$$

(3.4)

Following (3.4), we obtained a uniform order $p=\begin{bmatrix}6 & 6 & 6 & 6 & 6\end{bmatrix}^T$ for (2.6), whose error constants were calculated as $c_{p+1}=(-9.79(03) \quad 4.48(03) \quad -3.16(03) \quad 1.32(03) \quad -1.30(03))^T$.

C. Order of the block ETRs for (2.7)

Applying (1.5) on (2.7), we obtained

$$L\{y(x);h\}=\begin{bmatrix}1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&0&1&0&0&0\\0&0&0&0&1&0&0\\0&0&0&0&0&1&0\\0&0&0&0&0&0&1\end{bmatrix}\begin{bmatrix}y_{n+1}\\y_{n+2}\\y_{n+3}\\y_{n+4}\\y_{n+5}\\y_{n+6}\\y_{n+7}\end{bmatrix}-\begin{bmatrix}0&0&0&1&0&0&0\\0&0&0&1&0&0&0\\1&0&0&0&0&0&0\\0&0&0&1&0&0&0\\0&0&0&1&0&0&0\\0&0&0&1&0&0&0\\0&0&0&1&0&0&0\end{bmatrix}\begin{bmatrix}y_{n}\\y_{n+1}\\y_{n+2}\\y_{n+3}\\y_{n+4}\\y_{n+5}\\y_{n+6}\end{bmatrix}$$

$$-\begin{bmatrix}\frac{32}{3780}&-\frac{1363}{3780}&-\frac{4968}{3780}&-\frac{1153}{3780}&-\frac{208}{3780}&\frac{135}{3780}&-\frac{40}{3780}&\frac{5}{3780}\\-\frac{135}{120960}&\frac{4183}{120960}&-\frac{57627}{120960}&-\frac{81693}{120960}&\frac{20227}{120960}&-\frac{7227}{120960}&\frac{1719}{120960}&-\frac{191}{120960}\\\frac{1325}{4480}&\frac{6795}{4480}&\frac{1377}{4480}&\frac{5927}{4480}&-\frac{3033}{4480}&\frac{1377}{4480}&-\frac{373}{4480}&\frac{45}{4480}\\-\frac{191}{120960}&\frac{1879}{120960}&-\frac{9531}{120960}&\frac{68323}{120960}&\frac{68323}{120960}&-\frac{9531}{120960}&\frac{1879}{120960}&-\frac{191}{120960}\\0&\frac{5}{3780}&-\frac{72}{3780}&\frac{1503}{3780}&\frac{4688}{3780}&\frac{1503}{3780}&-\frac{72}{3780}&\frac{5}{3780}\\-\frac{13}{4480}&\frac{117}{4480}&-\frac{513}{4480}&\frac{2777}{4480}&\frac{3897}{4480}&\frac{5535}{4480}&\frac{1685}{4480}&-\frac{45}{4480}\\\frac{8}{945}&-\frac{64}{945}&\frac{216}{945}&-\frac{106}{945}&\frac{1784}{945}&\frac{216}{945}&\frac{1448}{945}&\frac{278}{945}\end{bmatrix}\begin{bmatrix}f_n\\f_{n+1}\\f_{n+2}\\f_{n+3}\\f_{n+4}\\f_{n+5}\\f_{n+6}\\f_{n+7}\end{bmatrix} \quad (3.5)$$

Expanding (3.5) in Taylor Series gives

$$\sum_{j=0}^{\infty}\frac{(h)^j}{j!}y_n^j - y_n^{j+3} - \frac{32h}{3780}y_n' - \sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\begin{Bmatrix} -\frac{1363}{3780}(1)^j-\frac{4968}{3780}(2)^j-\frac{1153}{3780}(3)^j \\ -\frac{208}{3780}(4)^j+\frac{135}{3780}(5)^j-\frac{40}{3780}(6)^j \\ +\frac{5}{3780}(7)^j \end{Bmatrix}$$

$$\sum_{j=0}^{\infty}\frac{(2h)^j}{j!}y_n^j - y_n^{j+3} + \frac{351h}{120960}y_n' - \sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left\{\begin{array}{l}\frac{4183}{120960}(1)^j - \frac{57627}{120960}(2)^j - \frac{81698}{120960}(3)^j \\ +\frac{20227}{120960}(4)^j - \frac{7227}{120960}(5)^j + \frac{1719}{120960}(6)^j \\ -\frac{191}{120960}(7)^j\end{array}\right\}$$

$$\sum_{j=0}^{\infty}\frac{(3h)^j}{j!}y_n^j - y_n - \frac{1325h}{4480}y_n' - \sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left\{\begin{array}{l}\frac{67953}{4480}(1)^j + \frac{1377}{4480}(2)^j + \frac{5927}{4480}(3)^j \\ -\frac{3033}{4480}(4)^j + \frac{1377}{4480}(5)^j - \frac{373}{4480}(6)^j \\ +\frac{45}{4480}(7)^j\end{array}\right\}$$

$$\sum_{j=0}^{\infty}\frac{(4h)^j}{j!}y_n^j - y_n^{j+3} + \frac{1916h}{120960}y_n' - \sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left\{\begin{array}{l}\frac{1879}{120960}(1)^j - \frac{9531}{120960}(2)^j + \frac{68323}{120960}(3)^j \\ +\frac{68323}{120960}(4)^j - \frac{9531}{120960}(5)^j + \frac{1879}{120960}(6)^j \\ -\frac{191}{120960}(7)^j\end{array}\right\}$$

$$\sum_{j=0}^{\infty}\frac{(5h)^j}{j!}y_n^j - y_n^{j+3} - \sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left\{\begin{array}{l}\frac{5}{3780}(1)^j - \frac{72}{3780}(2)^j + \frac{1503}{3780}(3)^j \\ +\frac{4688}{3780}(4)^j + \frac{1503}{3780}(5)^j - \frac{72}{3780}(6)^j \\ +\frac{5}{3780}(7)^j\end{array}\right\}$$

$$\sum_{j=0}^{\infty}\frac{(6h)^j}{j!}y_n^j - y_n^{j+3} + \frac{13h}{4480}y_n' - \sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left\{\begin{array}{l}\frac{117}{4480}(1)^j - \frac{513}{4480}(2)^j + \frac{2777}{4480}(3)^j \\ +\frac{3897}{4480}(4)^j + \frac{5535}{4480}(5)^j + \frac{1685}{4480}(6)^j \\ -\frac{45}{4480}(7)^j\end{array}\right\}$$

$$\sum_{j=0}^{\infty}\frac{(7h)^j}{j!}y_n^j - y_n^{j+3} - \frac{8h}{945}y_n' - \sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left\{\begin{array}{l}-\frac{64}{945}(1)^j + \frac{216}{945}(2)^j - \frac{106}{945}(3)^j \\ +\frac{1784}{945}(4)^j + \frac{216}{945}(5)^j + \frac{1448}{945}(6)^j \\ +\frac{278}{945}(7)^j\end{array}\right\}$$

$$=\begin{bmatrix}0 & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}^T \qquad (3.6)$$

From (3.6), we get that equation (2.7) has uniform order $p = [8 \quad 8 \quad 8 \quad 8 \quad 8 \quad 8 \quad 8]^T$ with error constants as $c_{p+1} = (-8.24(03) \quad 1.12(03) \quad -8.91(04) \quad 6.88(04) \quad -2.03(04) \quad 1.81(03) \quad -7.55(03))^T$.

The block integrators (2.5), (2.6) and (2.7) were found to be zero-stable and consistent, hence convergent.

4. Application to the systems of first order differential equations

In this section, we implement our proposed block ETRs method for step numbers $k = 3,5$ and 7 on two first order initial value problems.

Example 1: Let us consider the following system of ordinary differential equations on the range $x \in [0,10]$ highlighted by Okunuga *et al.,* 2009.

$$\frac{dy_1}{dx} = -0.5y_1, \qquad y_1(0) = 4$$

$$\frac{dy_2}{dx} = 4 - 0.3y_2 - 0.1y_1, \qquad y_2(0) = 6$$

The theoretical solution for the system is given as

$$y_1(x) = 4e^{-0.5x} \quad \text{and} \quad y_2(x) = \frac{-4e^{-\frac{x}{2}} + 40}{3} - 6$$

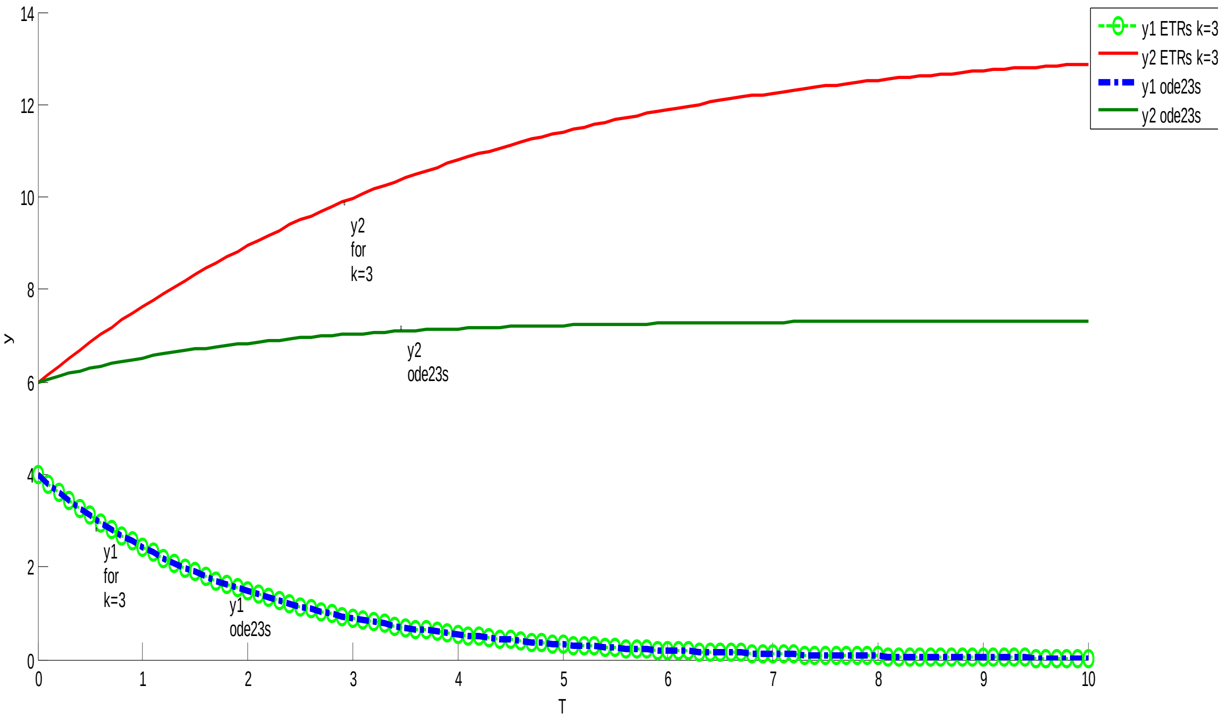

Fig. 1: Implementation of ETRs $k = 3$ on Example 1.

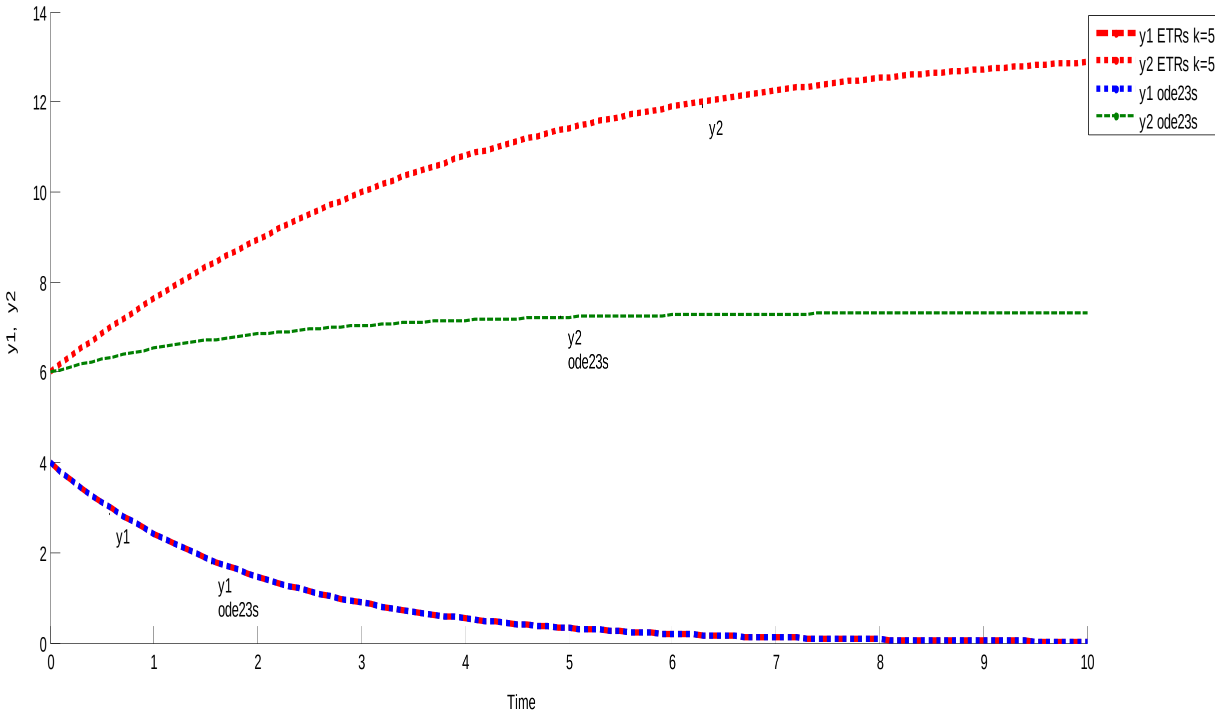

Fig. 2: Implementation of ETRs $k = 5$ on Example 1.

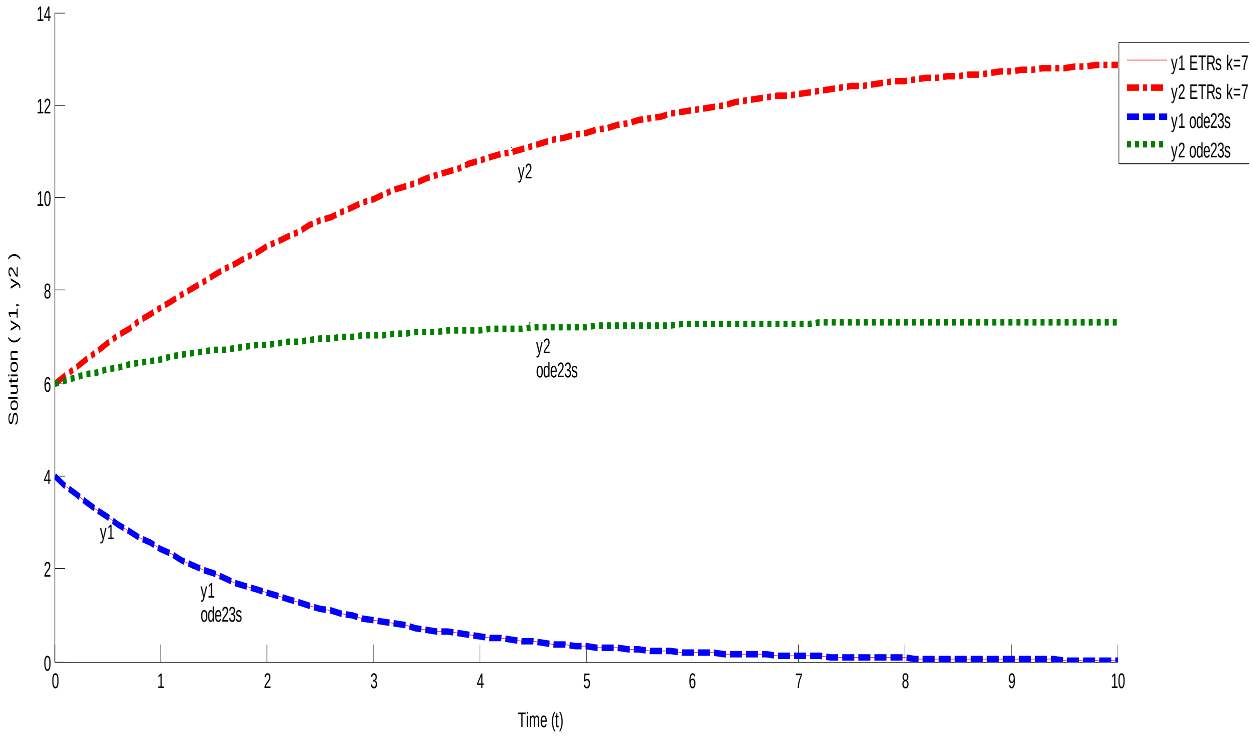

Fig. 3: Implementation of ETRs $k = 7$ on Example 1.

Example 2: Consider the linear system of ordinary differential equations reported by Riaz *et al.*, 1985.

$$\frac{dy_1}{dx} = -15.5y_1 + 14.5y_2 - 13.5, \quad y_1(0) = 3$$

$$\frac{dy_2}{dx} = 14.5y_1 - 15.5y_2 + 16.5, \quad y_2(0) = 2, \quad x \in [0,4]$$

The theoretical solution for the system is given by

$$y_1(x) = e^{-x} + e^{-30x} + 1 \quad \text{and} \quad y_2(x) = e^{-x} - e^{-30x} + 2$$

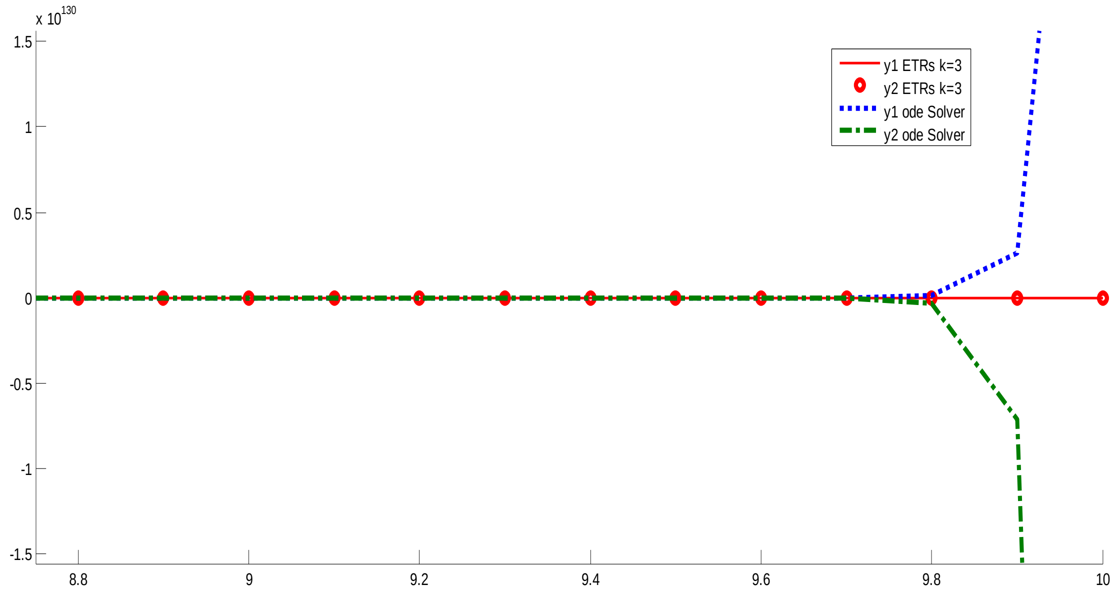

Fig. 4: Implementation of ETRs $k = 3$ on Example 2.

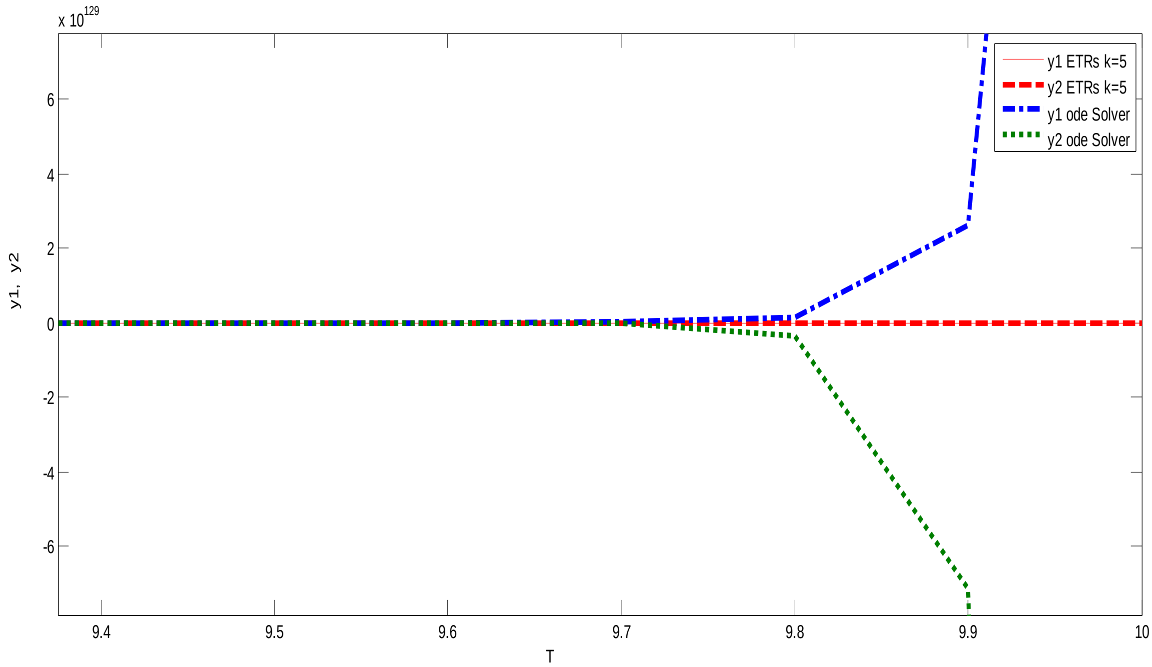

Fig. 5: Implementation of ETRs $k = 5$ on Example 2.

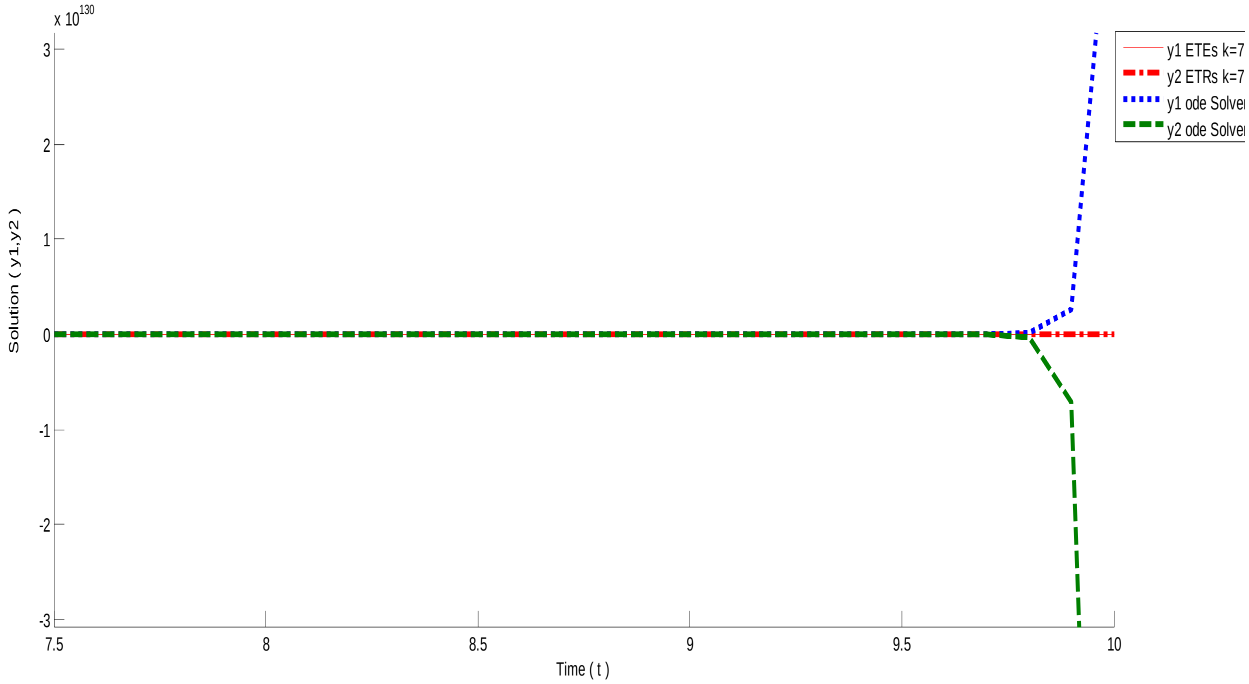

Fig. 6: Implementation of ETRs $k = 7$ on Example 2.

5. Conclusion

The beauty of our block ETRs methods for the specified step number have been shown when implemented on examples 1 and 2. Our method tend to competes well with the standard ode solver (for sake of reporting, see fig 1-6). The three step block ETRs method has been shown to possess the A-stability property of the trapezoidal rule in this class of method (see fig.7).

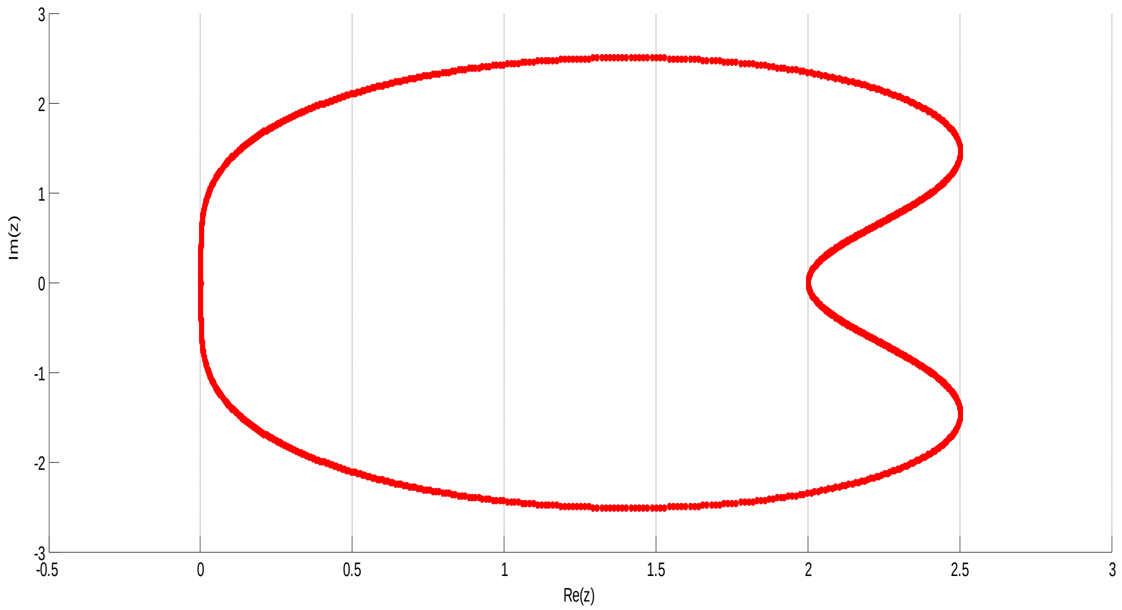

Fig. 7: Stability Region of the Three Step Block ETRs method.

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