# DETERMINATION OF THE NUMBER OF NON-ABELIAN ISOMORPHIC TYPES OF CERTAIN FINITE GROUPS

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## DECLARATION

I hereby declare that this work is the product of my own research efforts, undertaken under the supervision of Professor M.S. Audu and has not been presented elsewhere for the award of a degree or certificate. All sources have been duly distinguished and appropriately acknowledged.

**……………………………………**
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## CERTIFICATION

This is to certify that the research work for this thesis, and the subsequent preparation of this thesis by Martin Chuks Obi (PG/NS/UJ/0068/04) was carried out under my supervision.

**Sign:……………………………..** **……………………………..**

**Professor M.S. Audu** **Date**
*(Supervisor)*

**Sign:…………………………** **……………………………….**

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*(Head of Department)*

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## DEDICATION

This work is dedicated to my wife Mrs. Obi Angela Ojinika, our two daughters Mary Chidimma and Prisca Chizubelu and to the memory of my late daughter Rose Chika.

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| | |
|---|---|
| **$\pi$** | rotation through $180^0$ |
| ***** | binary operation of composition of rotation and reflections. |
| $\times$ | ordinary multiplication of numbers. |
| $\rightarrow$ | one-to-one correspondence map |
| $\varphi : G \rightarrow H$ | Group G is homomorphic to group H |
| **a\|b** | a divides b |
| **(p,q)** | the greatest common divisors of integers p and q |
| $\oplus$ | direct sum of groups |
| $\cap$ | intersection of groups |
| $\in$ | belongs to |
| **1** | Identity element |
| **$a \equiv b$** | a is congruence to b |
| **mod.** | Modulo |
| $\times \omega$ | multiplication of group elements induced by $\omega$ |
| **$\omega$** | an action induced by $a^{\omega}b=a^{-1}ba=b^{r}$ |
| $\equiv$ | not congruent |
| < | less than |
| > | Greater than |
| $\cong$ | isomorphism of two groups |
| $\langle a \rangle$ | group generated by an element a |
| **$N \lhd G$** | N is a normal subgroup of G |
| **G/K** | quotient group |

## LIST OF TABLES

# Abstract

The first part of this work established, with examples, the fact that there are more than one non-abelian isomorphic types of groups of order $n = sp$, $(s,p) = 1$, where $s<p$ and $p \equiv 1 \pmod{s}$ for $100 < p < 4000$. The factors s and p are distinct primes. Specifically considered here are groups of order $n = 2p, 3p, 5p, 7p, 11p$ and $13p$. It was discovered that the number of non-abelian isomorphic types of groups of order $n = sp$, $s<p$ increased as n increased. The defining relations of such non-abelian isomorphic groups were outlined and a scheme developed to generate the numbers for the non-abelian isomorphic types of such groups. The scheme helped in generating many examples of non-abelian isomorphic types of such groups. The situation where $p \equiv k \pmod{s}$, $k > 1$ was worked out and such groups have no non-abelian isomorphic types. This gave credence to the fact that a group of order 15 and its like do not have a non-abelian isomorphic type. It also generated the non-abelian isomorphic types of groups of order $n = spq$, where s, p and q are distinct primes considering the congruence relationships between the primes. It was seen that there are more non-abelian isomorphic types when $q \equiv 1 \pmod{p}$, $q \equiv 1 \pmod{s}$ and $p \equiv 1 \pmod{s}$. When q is not congruent to 1 modulo p but congruent to 1 modulo s fewer non-abelian isomorphic types were obtained. Moreover, if q is not congruent to 1 modulo p, q not congruent to 1 modulo s, and p not congruent to 1 modulo s, there cannot be a non-abelian isomorphic type of a group of order $n = spq$. In this case groups of order $n = 2pq, 3pq, 5pq$ and $7pq$ were considered. Later, proofs of the number of non-abelian isomorphic types for $n = sp$ and $n = spq$ using the examples earlier generated were given.

# CHAPTER ONE
# INTRODUCTION

## 1.0 BACKGROUND OF STUDY

Group Theory is relevant to every branch of Mathematics where symmetry is studied. Every symmetrical object is associated with a group. It is due to this association that groups arise in different areas like Quantum Mechanics, Crystallography, Biology, and even Computer Science. There is no such easy definition of symmetry among objects without leading its way to the theory of groups. Classifying groups arise when trying to distinguish the number of isomorphic groups of order n. In organic chemistry, conformal factors affect the structure of a molecule and its physical, chemical and biological properties. For instance, the position of atoms, relative to one another affects the structural formula of Hydrogen peroxide, $H_2O_2$. We could write two different planar geometries that differ by a $180^0$ rotation about 0 – 0 bond. According to Francis A Carey (2003) one could also write an infinite number of non planar structures by tiny increments of rotation about the 0 – 0 bonds; Francis A Carey (2003). Groups may be presented in several ways like multiplication table, by its generators and relations, by Cayley graph, as a group of transformations (usually a geometric object), as a subgroup of a permutation group, or a subgroup of a matrix group to mention a few.

## 1.1 STATEMENT OF THE PROBLEM

Classifying groups arise when trying to distinguish the number of isomorphic types of a group of order n.

Hall Jnr and Senior (1964) used invariants as the number of elements of each order k (k small) to determine whether two groups of order $2^n$ ($n \leq 6$) are isomorphic. Philip (1988) in his article developed a systematic classification theory for groups of prime power orders. For certain classes of groups, there exists practical methods to list such groups. Newman and O'Brien (1990) introduced an algorithm to determine up to isomorphism the groups of prime-power order. The determination of all groups of a given order up to isomorphism is an old question in group theory. It was introduced by Cayley who constructed the groups of order 4 and 6 in 1854.

Meubüser (1967) listed all groups of order at most 100 except for 64 and 96. The groups of order 96 were added by Lane (1982).

Moreover, for factorizations of certain orders, the corresponding groups have been classified, e.g. Holder (1983) determined the groups of order $pq^2$ and pqr, and James (1980) determined the groups of order $p^n$ for odd primes and $n \leq 6$.

Recently, algorithms have been used to determine certain groups. For example O'Brian (1991) determined the 2-groups of order at most $2^8$ and the 3-groups of order at most $3^6$. Moreover, Betten (1996) developed a method to construct finite soluble groups and used his construction to construct soluble groups of order at most 242.

Determination of isomorphic types has been a comparatively difficult problem as there was no method that is sufficiently effective.

Most of the classifications of the non-abelian isomorphic types of certain finite groups were done for groups of small orders. This is possibly due to the complexity of computation as the factors increase. The problem then arise to find the non-abelian isomorphic types of groups of higher orders which can be factorized into two or three distinct primes taking into consideration of the relationship between the prime factors. The need also arise to construct a suitable computer program to assist in solving such a problem.

Hence, the statement of the problem is "Determination of the Number of non-Abelian Isomorphic Types of Certain Finite Groups".

## 1.2 AIM AND OBJECTIVES

The aim of this thesis is to determine the number of non-abelian isomorphic types of certain finite groups of higher orders.

We hope to achieve the following objectives:

(i) Finding relationship, through series of examples, of the number of non-Abelian Isomorphic types of groups of order n=sp and the congruence relation between the primes s and p.

(ii) Determining the proof for the number of non-Abelian isomorphic types in each congruence relationship and stating their defining relations.

(iii) Determine and design a suitable computer program that will help in working out the number relationship between such primes and generating the numbers for the non-Abelian isomorphic types.

(iv) Finding the non-Abelian isomorphic types of groups of order n = spq where s,p and q are distinct primes and determining their defining relations.

## 1.3 SCOPE OF THE STUDY

The scope here is limited to the determination of the number of non-Abelian isomorphic types of groups of order 2p, 3p, 5p, 7p, 11p, 13p where $p < 4000$. Also considered are groups of order 2pq, 3pq, 5pq and 7pq. The primes p and q are distinct primes with $p < q$.

## 1.4 DEFINITION OF THE CONCEPT OF ISOMORPHIC GROUPS

The concept of group isomorphism can be explained with chessboard that has four plane symmetries. The identity, rotation r through $\pi$ about its centre, and the reflections $q_1, q_2$ in its two diagonals form a group under composition whose multiplication is given in table 1 below.

**Table 1.1:** Four plane symmetries of a chessboard

| $*$ | $e$ | $r$ | $q_1$ | $q_2$ |
|---|---|---|---|---|
| $e$ | $e$ | $r$ | $q_1$ | $q_2$ |
| $r$ | $r$ | $e$ | $q_2$ | $q_1$ |
| $q_1$ | $q_1$ | $q_2$ | $e$ | $r$ |
| $q_2$ | $q_2$ | $q_1$ | $r$ | $e$ |

It is easy to check that multiplication modulo eight makes the numbers 1,3,5,7 into a group.

There is an apparent similarity between these two groups if we ignore their origins. In each case the group has four elements, and these elements appear to combine in the same manner. Only the way in which the elements are labeled distinguishes one table from the other.

Label the first group G, the second G', and the correspondence.

$e \rightarrow 1, r \rightarrow 3, q_1 \rightarrow 5, q_2 \rightarrow 7$ ,

This correspondence is called an isomorphism between G and G'. It is a bijection and it carries the multiplication of G to that of G'. Technically they are isomorphic in the following sense.

Two groups G and G' are isomorphic if there is a bijection $\varphi$ from G to G' which satisfies $\varphi(xy) = \varphi(x)\varphi(y)$ for all $x, y \in G$. The function $\varphi$ is called an isomorphism between G and G'.

Hence the isomorphism as a bijection implies that G and G' must have the same order. It sends the identity of G to that of G'. Isomorphism also preserves the order of each element (Armstrong, 1988).

### 1.4.1 EXAMPLES

(i) The group of all real numbers with addition, (R,+), is isomorphic to the group of all positive real numbers with multiplication $(R^+, \times)$.

Proof: Define $f : (R,+) \to (R^+, \times)$ by $f(x) = e^x$. For elements x, y in R then

$f(x) = f(y)$ then $e^x = e^y$, so $x = y$. This implies that $x \neq y$, then $f(x) \neq f(y)$ i.e., $e^x \neq e^y$.

If r is an element o $R^+$, then f(ln r) = r, where ln r belong to R showing that f is onto $R^+$. Again, for elements x, y in R, we have

$$f(x + y) = e^{x+y} = e^x.e^y = f(x)f(y).$$

Hence (R,+) is isomorphic to $(R^+, \times)$.

(ii) Every cyclic group of infinite order is isomorphic to the additive group I of integers

Proof: Consider the infinite cyclic group G generated by a and the mapping

$n \to a^n$, $n \in I$ of I into G.

Now, this mapping is onto since any n in I is mapped to exactly one $a^n$.

Moreover, it is one-to one since if $s > t$ we have $s \leftrightarrow a^s$ and $t \leftrightarrow a^t$, then

$a^{s-t} = 1$ and G would be finite. Hence if s ‡ t, then $a^s \neq a^t$.

Finally, $s + t \leftrightarrow a^{s+t} = a^s.a^t$. Hence the mapping is an isomorphism, that is

$I \cong G$.

(iii) The group Z of integers (with addition) is a subgroup of R, and the factor group R/Z is isomorphic to the group S' of complex numbers of absolute value 1 (with multiplication):

$R/Z \cong S'$

An isomorphism is given by

$f(x+Z) = e^{2\pi xi}$

for every x in R.

Proof: We only need to show that for any $k \in Z$, then $f(x + k) = e^{2\pi i(x + k)} = e^{2\pi xi + 2i\pi k}$

$= e^{2\pi xi}. e^{2\pi ki} = e^{2\pi xi}(\cos 2\pi k + i\sin 2\pi k) = e^{2\pi xi}$.

If $x \neq y$ then $f(x + k) \neq f(y + k)$, i.e., $e^{2\pi xi} \neq e^{2\pi yi}$. Also for $z \in R$, then $f(\ln(z + k))$

$= e^{\ln(z + k)} = z + k$.

(iv) The Klein four-group is isomorphic to the direct product of two copies of $Z_2 = Z/2Z$ and can therefore be written $Z_2xZ_2$. Another notation is $D_2$, because it is a dihedral group.

(v) Generalizing this, for all odd n, $D_{2n}$ is isomorphic with the direct product of $D_n$ and $Z_2$.

### 1.4.2 PROPERTIES OF ISOMORPHIC GROUPS

(i) The Kernel of an isomorphism from $(G,*)$ to $(H, \Theta)$, is always $\{e_G\}$ where $e_G$ is the identity of the group $(G,*)$.

(ii) If $(G,*)$ is isomorphic to $(H,\Theta)$, and if G is Abelian then so is H.

(iii) If $(G,*)$ is a group that is isomorphic to $(H,\Theta)$ [where f is the isomorphism], then if a belongs to G and has order n, then so does f(a)

(iv) If $(G,*)$ is a locally finite group that is isomorphic to $(H,\Theta)$, then $(H,\Theta)$, is also locally finite.

We state mostly without proof certain fundamental results of group theory which we shall be needed:

### 1.4.3 THEOREM (LAGRANGE'S THEOREM)

Let G be a group of finite order n, and H a subgroup of G. The order of H divides the order of G.

### 1.4.4 THEOREM (CAUCHY'S THEOREM)

If p is a prime number and $p \| G |$ then G has an element of order p.

### 1.4.5 THEOREM (SYLOW'S FIRST THEOREM)

If $p^a$ is the highest power of a prime dividing the order of a group G, then G has at least one subgroup of order p

### 1.4.6 DEFINITION 1

For any prime, p, we say that a group G is a p-group if every element $x$ in G has order $p^k$, for some integer k

### 1.4.7 DEFINITION 2

Let G be a finite group of order n = pq, where (p,q) = 1. Then any subgroup of order $p^m$ is called a Sylow p-subgroup of G.

### 1.4.8. DEFINITION 3

Let a be an element of a group G and e the identity element of G. The smallest positive integer n such that $a^n = e$ is called the order of a. The order of a group G, written $|G|$ is the cardinal number of elements of G. G is said to be finite or infinite according as its order is finite or infinite (Kuku, 1980).

### 1.4.9 DEFINITION 4

Let G be a group and let a and b be elements of G then G contains both $\langle a \rangle$ and $\langle b \rangle$.

Other elements of G depends on the relation between a and b. The smallest subgroup generated by a anb b is denoted by $\langle a, b \rangle$. If ab = ba then G is said to be Abelian or commutative. If $ab \neq ba$ then G is said to be non-Abelian or is said to be not commutative.

**1.4.10 THEOREM (SYLOW'S SECOND THEOREM)**

All Sylow p-subgroups of a finite group G belonging to the same prime are conjugate with one another in G.

**1.4.11 THEOREM (SYLOW'S THIRD THEOREM)**

Let r be the number of Sylow p-subgroups of G, then r is an integer of the form 1+kp and r is a factor of the order of G.

**1.4.12 THEOREM (A BASIS THEOREM FOR FINITE ABELIAN GROUPS)**

Every finite Abelian group is a direct sum of primary cyclic groups.

**1.4.13 THEOREM (ANOTHER BASIS THEOREM FOR FINITE ABELIAN GROUPS)**

Every finite Abelian group A can be decomposed into a direct sum of cyclic groups.

$$A = C_{m_1} \oplus C_{m_2} \oplus .... \oplus C_{m_{s-1}}$$

Where $m_{1+1} | m_i$ for all i= 1,2,…,s-1

**1.4.14 THEOREM**

If H and K are normal subgroups of G such that $H \cap K = \{1\}$ then any element $x$ of H commutes with any other element y of K.

**PROOF:**

For any $x \in H$ and $y \in K$, consider the commentator

$z = xyx^{-1}y^{-1} = (xyx^{-1})y^{-1} = x(yx^{-1}y^{-1})$ and notice from the first factorization and the normality of K that $y^{-1} \in K$, $xyx^{-1} \in K \Rightarrow z \in K$.

Furthermore, since H is normal, we have from the second factorization that

$x \in H, yx^{-1}y^{-1} \in H \Rightarrow z \in H$

Hence, we deduce that

$$z \in H \cap K = \{1\} \Rightarrow z = 1$$

Whence

$xy = yx$ as asserted

**1.4.15 PROPOSITION**

Let G be a finite group and K any normal subgroup contained in the centre of the group G. Then if G is non-Abelian the quotient group G/K cannot be cyclic.

**PROOF:**

Suppose

$$G / K = \{K, tK, ..., t^{n-1}K\}$$

Then for any $x, y \in G$ we have

$$x = t^{s}u, y = t^{r}v,$$

For some $u, v \in K$ and thus

$$xy = t^{s}ut^{r}v = t^{k+r}uv = t^{r+s}vu = t^{r}vt^{s}u = yx$$

(Since u, v permute with t).

This contradicts the non-Abelian hypothesis on G.

The problem of explicitly constructing all the groups of a given finite order has a long and somewhat chequered history; its study was initiated by Cayley in 1864 when he determined the groups of order at most 6. The aim is to determine a complete and irredundant list of the groups of a given order: a representative of each isomorphism type present. It is usually comparatively easy to generate a complete list; the difficulty lies in the reduction to distinct isomorphism types (Hall M., 1976).

**1.4.16 THEOREM (FROBENIUS)**

Let H be a p-subgroup of order $p^a$ in G. Let K, of order $p^b$, be the intersection of H and some other p-Sylow subgroup H' of G such that no subgroup of G containing K and of order greater than $p^b$ is contained in any two p-Sylow subgroups. Then G must contain an element of order prime to p which permutes with K but does not permute with H.

**1.4.17 REMARK**

a) The subgroup K is a subgroup of maximum order common to both H and H', it does not necessarily have maximal order among the intersections of any two p-Sylow subgroups.

b) There is a parallel theorem when p-Sylow subgroups H and H' are both Abelian. In this case, every element of K is self-conjugate in the subgroup gp{H,H'}.

Thus if N is the greatest subgroup of G in which every element of K is self-conjugate, then N contains two and hence

$1 + kp$ p-Sylow subgroups.

That is, N has order $p^a m'(1+kp)$ where $p^a m'$ is the order of the greatest subgroup of the normalizer of H (of order $p^a m$) in which every element of K is self-conjugate. Thus, in this case, there is an element of order p which permutes with every element of K.

**1.4.18 POLYNOMIAL:**

A function of z of the form

$P(z) = a_0 + a_1z + a^2z^2 + \ldots + a_nz^{n,}$

in which $a_n \neq 0$ is called a polynomial of degree n in z.

**1.4.19 THEOREM**

Every polynomial of degree n (where $n \geq 0$) has at least one root and at most n roots (Mervin 1986).

# CHAPTER TWO
# LITERATURE REVIEW

## 2.0 MULTIPLICATION TABLES OF GROUPS OF ORDER 2 TO 10.

In his work, Wavrik J. (2002) developed a JAVA applet that allows experimentation with group multiplication tables. Here we present some of his work for groups of order 6 and 10. It was noted that any group of order 6 and 10 is isomorphic to one of the groups given below and some their tables are outlined in Tables 2.1 and 2.2 below.

**$C_6$, the cyclic group of order 6**

Described via the generator a

with relation $a^6 = 1$:

Elements:

Order 6: a, $a^5$

Order 3: $a^2$, $a^4$

Subgroups:

Order 6: $\{1, a, a^2, a^3, a^4, a^5\}$

Order 3: $\{1, a^2, a^4\}$

Order 2: $\{1, a^3\}$

Order 1: $\{1\}$

**$S_3$, the symmetric group on three elements**

Described via generator a, b

with relations $a^3 = 1$, $b^2 = 1$, $ba = a^{-1}b$:

Elements:

Order 3: a, $a^2$

Order 2: b, ab, $a^2b$

Subgroups:

Order 6: {1, a, $a^2$, b, ab, $a^2b$}

Order 3: {1, a, $a^2$}

Order 2: {1, b} {1, ab} {1, $a^2b$}

Order 1: {1}

Normal subgroups:

Order 6: {1, a, $a^2$, b, ab, $a^2b$

Order 3: {1, a, $a^2$}

Order 1: {1}**Table 2.1:** Symmetric group of order 6

| X | 1 | a | $a^2$ | b | ab | $a^2b$ |
|---|---|---|---|---|---|---|
| 1 | 1 | a | $a^2$ | b | ab | $a^2b$ |
| a | a | $^2$ | 1 | ab | $a^2b$ | b |
| $a^2$ | $a^2$ | 1 | a | $a^2b$ | b | ab |
| b | a | $a^2b$ | ab | 1 | $a^2$ | a |
| ab | ab | b | $a^2b$ | a | 1 | $a^2$ |
| $a^2b$ | $a^2b$ | ab | b | $a^2$ | a | 1 |

**$C_{10}$, the cyclic group of order 10**

Described via the generator a with relation $a^{10} = 1$:

Elements:

Order 10: a, $a^3$, $a^7$, $a^9$

Order 5: $a^2$, $a^4$, $a^6$, $a^8$

Order 2: $a^5$

Subgroups:

Order 10: $\{1, a, a^2, a^3, a^4, a^5, a^6, a^7, a^8, a^9\}$

Order 5: $\{1, a^2, a^4, a^6, a^8\}$

Order 2: $\{1, a^5\}$

Order 1: $\{1\}$

**$D_5$, the dihedral group of order ten**

Described via generators a, b

With relations $a^5 = 1$, $b^2 = 1$, $ba = a^{-1}b$:

Elements:

Order 5:a, $a^2$, $a^3$, $a^4$

Order 2: b, ab, $a^2b$, $a^3b$, $a^4b$

Subgroups:

Order 10: $\{1, a, a^2, a^3, a^4, b, ab, a^2b, a^3b, a^4b\}$

Order 5: $\{1, a, a^2, a^3, a^4\}$

Order 2: $\{1, b\}$, $\{1, ab\}$, $\{1, a^2b\}$, $\{1, a^3b\}$, $\{1, a^4b\}$

Order 1: $\{1\}$

Normal subgroups:

Order 10: $\{1, a, a^2, a^3, a^4, b, ab, a^2b, a^3b, a^4b\}$

Order 5: $\{1, a^2, a^3, a^4\}$

Order 1: $\{1\}$.

**Table 2.2:** Symmetric group of order 10, $S_5$

| X | 1 | a | $a^2$ | $a^3$ | $a^4$ | b | ab | $a^2b$ | $a^3b$ | $a^4b$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | a | $a^2$ | $a^3$ | $a^4$ | b | ab | $a^2b$ | $a^3b$ | $a^4b$ |
| a | a | $a^2$ | $a^3$ | $a^4$ | 1 | ab | $a^2b$ | $a^3b$ | $a^4b$ | b |
| $a^2$ | $a^2$ | $a^3$ | $a^4$ | 1 | a | $a^2b$ | $a^3b$ | $a^4b$ | b | ab |
| $a^3$ | $a^3$ | $a^4$ | 1 | a | $a^2$ | $a^3b$ | $a^4b$ | b | ab | $a^2b$ |
| $a^4$ | $a^4$ | 1 | a | $a^2$ | $a^3$ | $a^4b$ | b | ab | $a^2b$ | $a^3b$ |
| b | b | $a^4b$ | $a^3b$ | $a^2b$ | ab | 1 | $a^4$ | $a^3$ | $a^2$ | a |
| ab | ab | b | $a^4b$ | $a^3b$ | $a^2b$ | a | 1 | $a^4$ | $a^3$ | $a^2$ |
| $a^2b$ | $a^2b$ | ab | a | $a^4b$ | $a^3b$ | $a^2$ | a | 1 | $a^4$ | $a^3$ |
| $a^3b$ | $a^3b$ | $a^2b$ | ab | b | $a^4b$ | $a^3$ | $a^2$ | a | 1 | $a^4$ |
| $a^4b$ | $a^4b$ | $a^3b$ | $a^2b$ | ab | b | $a^4$ | $a^3$ | $a^2$ | b | 1 |

We now give results on group classification up to isomorphism which are basic to this work.

John R. Durbin (1979) showed the number of isomorphic types of groups of order n for each n from 1 to 32 and stated as follows: "There is just one group of order n if and only if n is a product of distinct primes $p_1, p_2, ..., p_k$ such that $p_j \nmid (p_i - 1)$ for $1 \leq i \leq k, 1 \leq j \leq k$".

The above conclusion was reached using groups of orders 15 = 3 x 5, 33 = 3 x 11.

## 2.1 ISOMORPHIC TYPES OF GROUPS OF ORDER n = pq

Let G be a group of order n = pq, where p and q are distinct primes with p < q. Then by Sylow's theorem (1.4.5) there must be only one Sylow q-subgroup in G. This subgroup

$K = gp\{b\}, b^q = 1,$

and must be normal in G.

Moreover, any Sylow p-subgroup must be of the form

$$H = gp\{a\}, a^p = 1.$$

For $K \triangleleft G$, we must have

$a^{-1}ba \in K$ and

$a^{-1}ba = b^t$

for some integer t.

Clearly, if t = 1 we have that G in Abelian and so must be of order pq.

By (1.4.11) there is only one Sylow p subgroup and we have the cyclic group situation.

Suppose $t \neq 1$ then

$$a^{-1}b^k a = \left(a^{-1}ba\right)^k = b^{t^k}$$

$$a^{-2}b\ a^2 = a^{-1}\left(a^{-1}ba\right)a = b^{t^2}$$

This will be done up to some integer j such that

$$a^{-j}ba^j = b^{t^j}.$$

If j = p the following relations will be obtained since $a^p = 1$:

$$b = a^{-p}ba^p = b^{t^p}$$

We deduce that

$1 = b^{t^p - 1}$ and so

$$q \mid (t^p - 1) \Rightarrow t^p \equiv 1 (\text{mod } q)$$

The solutions of

$t^p \equiv 1(\text{mod } q)$ are $1, t, t^2, \ldots, t^p$,

and with the exception of 1, generate the same group, since the replacement of $H = gp\{a\}$ replace t by $t^j$

Conversely, if $t^p \equiv 1(\text{mod } q)$ and

$$ab = b^t a$$

then using the multiplication scheme:

$$a^u b^v \left(a^x b^y\right) = a^u a^x a^{-x} b^v a^x b^y$$

$$= a^{u+x} b^{vt^x + y}$$

$$= a^{u+x} b^{y+vt^x}$$

We obtain that G is the semi-direct product:

G =K $x_w$ H, where the action of $\omega$ is induced by

$$a^{\omega} b = a^{-1} ba = b^t$$

Thus the following is inferred from the above proof

## 2.2 PROPOSITION

There are at most two isomorphic types of groups of order pq, where p and q are distinct primes and p<q, namely:-

(i) The cyclic group of order pq and

(ii) The non-Abelian semi-direct product

$$gp\{b\} x_{\omega} gp\{a\}$$

where $a^p = b^q = 1, a^{-1}ba = b^t$

$t^p \equiv 1(\text{mod } q)$

$$t \not\equiv 1(\text{mod } q) \text{ and } p \mid (q-1)$$

(Michio, 1982).

For the group of order n = 2p, (2,p) =1, since any group of a prime order is necessarily cyclic, it is obvious that subgroups of G of orders 2 and p are cyclic. Hence G = $C_2$ x $C_p$. By Sylow's theorem (1.4.5) there must be only one $S_p$ -subgroup of G, $C_p$ say, such that Cp = gp{b}; $b^p = 1$.

This must be normal in G. Moreover, any other Sylow 2-subgroup will be of the form

$C_2 = \langle a \rangle; a^2=1.$

Hence $C_p \lhd G$ and $a^{-1}ba \in C_p$. We need to find the integer r such that $a^{-1}ba = b^t$ and $t \neq 1$. If such integer exists, then we have the non-Abelian group of order

$n = 2p$. But if there are p $S_2$ -subgroups of order 2 in G and only one Sp-subgroup, we will have a total of $p(2 - 1) + (p - 1) = p + p - 1 = 2p - 1$ elements in G excluding the identity element. This implies that there are p elements of order 2 in G some of which do not commute with the element b in $C_p$. Hence $a^{-1}ba = b^t$ for $t \neq 1$ and t is such that $t^2 \equiv 1 \pmod{p}$.

If we take values for t in the interval $1<t<p$, it is obvious that only one value will satisfy the congruence $t^2 \equiv 1 (\mathrm{mod}\ p)$ and this value gives the non-Abelian isomorphic types of the group G of order $n = 2p$.

From the above fact the following Corollary is stated:

## 2.3 COROLLARY

There is only one isomorphic class of a group of order 15 which is Abelian and two isomorphic types of groups of order 6, 10, 14, 21, 22, and 26, of which one is Abelian and the other is non-Abelian.

## 2.5 GROUPS OF ORDER $p^2q$

Let G be any group of order $p^2q$, where p and q are distinct. By a Basis Theorem for finite Abelian Groups which states that "Every finite Abelian group is a direct sum of primary cyclic groups. The isomorphism classes of Abelian groups of order $p^2q$ are given by the following invariants

$$p^2 \times q, \text{and } \ p \times p \times q.$$

The first form is cyclic and the second is not cyclic.

Suppose G is a n on-Abelian groups of order 12.

There are 1 or 4 Sylow 3 – subgroups of order 3 since if r is the number of Sylow p-subgroups of G, then r is an integer of the form 1+kp and r is a factor of the order of G.

For 4 Sylow 3– subgroups there will be 8 elements of order 3 leaving 4 elements which must constitute a unique 2 – Sylow subgroup and therefore normal in G. We claim in such a case there can be no element of order 4, x say; for otherwise, for some element a which is of order 3 we have, since ‹ x › is normal in G, that $a^{-1}xa = x$ or $x^3$, (the only powers of $x$ of order 4). But

$$a^{-1}xa= x$$

implies that G is Abelian, a contradiction.

Furthermore

$$a^{-1}xa= x^3$$

$$\Rightarrow a^{-2}xa^2=a^{-1}(a^{-1}xa)\, a=a^{-1}\text{-}\, x^3a=(x^3)^3= x^9= x.$$

Hence

$$x^3=a^{-1}xa=a^{-4}xa^4=a^{-2}xa^2= x,$$

which is absurd. Hence, we must have $a^{-1}xa= x$ and so G would necessarily be Abelian. Hence, we must have that the 2-Slow subgroup is of the Klein type, say

$K= \langle x,y \rangle$, $x^2=1$, $y^2=1$; $xy=y\,x$.

If z is any element of order 3 in G, it must permute the three elements of order 2 in K amongst themselves: that is we may set $z^{-1}xz=y$; $z^{-1}yz= xy$,

and obtain a single new type:

(iii) $G_3= \langle x,y,z \rangle$, $x^2=1, y^2=1, z^3=1; z^{-1}xz=y, z^{-1}yz= xy$.

Since the Sylow 2 – subgroup is normal if the Sylow 3-subgroup is not, it follows that if the $S_2$ – subgroup is not normal, then the 3-Sylow subgroup must be normal. Thus, we now assume that the 3-Sylow subgroup is unique, and hence normal by Sylow's Second Theorem which states that "all Sylow p-subgroups of a finite group G belonging to the same prime are conjugate with one another in G. We may thus consider

$$K = \langle a \rangle;\ a^3 = 1$$

and note that either the $S_2$ – subgroup is cyclic or it is non-cyclic since there are two isomorphic types of groups of order 4, 9 and 25.

in the former situation, we have an element b of order 4 and since G is non-Abelian we must have, by virtue of the normality of K,

$$b^{-1}ab = a^4 = a^{-1},$$

Moreover,

$$b^{-2}ab^2 = a^4 = a,$$

and deduce that $b^2$ commutes with a and the two together generate a cyclic subgroup of order 6. We therefore have the following isomorphic type:

(iv) $G_4 = \langle a,b \rangle, a^3=1, b^4=1; b^{-1}ab=a^{-1}$.

Suppose now that we have an $S_2$ – subgroup of the form

$$H = \langle b,c \rangle, b^2=1, c^2=1; bc=cb.$$

Then since G is non-Abelian at least one of b or c does not commute with a. Suppose

$$b^{-1}ab = a^2, \text{ then } c^{-1}ac = a \text{ or } c^{-1}ac = a^2.$$

In the former we have

$$(bc)^{-1}a(bc) = c^{-1}b^{-1}abc = a^2$$

Also if

$$c^{-1}ac = a^2, \text{ then}$$

$$(bc)^{-1}a(bc) = c^{-1}b^{-1}abc = c^{-1}a^2c = (a^2)^2 = a^4 = e.$$

Hence we have the following isomorphic type

(v) $G_5 = \langle a,b,c \rangle, a^3=1, b^2=1, c^2=1; bab = a^{-1}, bc = cb, ca = ac.$

$= \langle a,b \rangle \times \langle c \rangle = D_3 x C_2.$

If we set

$x = ac$, then $x^2 = a^2 = a^{-1}$, $x^3 = e$;

also we may set

$y = b$

and deduce that

$y\,xy = bacb = a^{-1}c = x^2$, $x^3 = y^5 = 1.$

Thus

$\langle a,b,c \rangle = \langle x,y \rangle, x^6=1, y^2=1; y\,xy = x^{-1}$

$\cong D_6,$

which is the dihedral group of order 12. Hence we have proved the following:

**2.5.1 PROPOSITION**

There are five isomorphism classes of groups of order 12, two are Abelian while the remaining three are non-Abelian (Okorie and Obi, 1991).

**2.5.2 SUMMARY OF DEFINING RELATIONS**

(i) $G_1 = \langle x \rangle, x^{12}=1.$

(ii) $G_2 = \langle x,y \rangle, x^6=1, y^2=1;\ xy = y\,x.$

(iii) $G_3 = \langle x,y,z \rangle, x^2=1, y^2=1, z^3=1; z^{-1}xz = y, z^{-1}yz = xy,\ xy = y\,x.$

(iv) $G_4 = \langle x,y \rangle, x^4=1, y^3=1;\ x^{-1y}x = y^{-1}.$

(v) $G_5 = \langle x,y \rangle, x^6=1, y^2 1; y\,xy = x^{-1}.$

**2.5.3 REMARK**

The group $G_3$ has no subgroup of order 6; this is the only class of groups of order 12 with this property and provides the first counter example to the converse of Lagrange's Theorem. Thus, it is no true in general that if m is a factor of n, then any group of order n has some subgroup of order m.

Next, suppose G is a non-Abelian group of order $18=3^2x2$.

By (1.4.11) there are 1, 3 or 9 Sylow 2 – subgroups. Also, we have exactly one 3 – Sylow subgroup. If there were only 1 Sylow 2-subgroup, then by Proposition (1.4.15) G would be a direct product of the form

$$C_2xH,$$

where H is the unique 3-Sylow subgroup of order $9=3^2$ which is Abelian. Thus, it follows that if G is non-Abelian we need consider cases in which the Sylow 3-subgroup is normal in G. If the subgroup K of order 9 were cyclic, we may present this subgroup by

$$K=\langle x\rangle,\ x^9=1.$$

Also by Cauchy's Theorem (1.4.4) we have some element y of order 2 in G.

Clearly, $y\notin K$.

Moreover, $K\triangleleft G$.

$$\Rightarrow y^{-1}Ky = K$$

and in particular, we have

$$y^{-1}xy= x^t, \text{ where}$$

$$x=y^{-2}= xy^2= x^{t^2} \text{ (since } y^2=1).$$

We deduce that

$$x^{t^2} = x$$

and hence

$$t^2\equiv 1 \text{ (mod. 9)}.$$

The solutions of this congruence relations in the range $1 \leq t < 9$ are 1 and 8. Since t =1 entails that G is Abelian, it follows that

$$y^{-1} xy = x^8 = x^{-1}$$

and we have

(iii) $G = \langle x, y \rangle,\ x^9 = 1, y^2 = 1; y x y = x^{-1}$.

The above class is the class of the dihedral group of order 18, $D_9$.

On the other hand let the 3-Sylow subgroup be of the form

$$N = \langle x, y \rangle\ x^3 = 1, y^3 = 1;\ xy = y x.$$

Then, since we have some element z of order 2, and since the non-Abelian nature of G forbids z commuting with both $x$ and y we must have the following possibilities:

(a) $xz = z x$, $zyz = y^{-1}$.

(b) $z x z = x^{-1}$, $zyz = y^{-1}$.

Thus we have the following classes

(iv) $G_4 = \langle x, y, z \rangle,\ x^3 = 1, z^2 = 1;\ xy = y x,\ z x = xz,\ zyz = y^{-1}$.

$= \langle x \rangle \text{ x } \langle, z \rangle$

$\cong C_3 x D_3$.

(v) $G_5 = \langle x, y, z \rangle,\ x^3 = 1, y^3 = 1, z^2 = 1;\ xy = y x, z x z = x^{-1}, zyz = y^{-1}$.

We have therefore proved the following

**2.5.4 PROPOSITION**

There are five classes of groups of order 18, two are abelian of which one is cyclic, and three are non-Abelian.

### 2.5.5 SUMMARY OF DEFINING RELATIONS

(i) $G_1 = \langle x \rangle, x^{18}=1.$

(ii) $G_2 = \langle x,y \rangle, x^9=1, y^2=1;\ xy=yx.$

(iii) $G_3 = \langle x,y \rangle, x^9=1, y^2=1; yxy= x^{-1}.$

(iv) $G_4 = \langle x,y,z \rangle, x^3=1, y^3=1, z^2=1;\ xy= x$

$zyz=y^{-1}, zx= xz.$

(v) $G_5 = \langle x,y,z \rangle, x^3=1, y^3=1, z^2=1;\ xy=yx, zy=y^{-1}, zxz= x^{-1}.$

Again, suppose G is a non-Abelian group of order $20=2^2 x5$.

By (1.4.11) there are exactly 1 Sylow 5-Subgroup,

$\langle x \rangle, y^5=1,$

$x^{-2}yx^2=y^4,$

$x^{--3}yx^{-3}=y^2.$

Since $x$ and $x^3$ are alternative generators of the 2-Sylow subgroup, it follows that the two sets of relations give rise to the same isomorphism class.

Moreover,

$yx= xy^4 \Rightarrow x^{-1}yx=y^{4,}$

$x^{-2}yx^2=y$

$x^{-3}yx^3=y^4.$

That is, in this case, $x^2$ commutes with y and we do obtain a different isomorphic type.

Hence we have the following isomorphic types.

(i) $G_3 = \langle x,y \rangle, x^4=1, y^5=1; yx= xy^2.$

(ii) $G_4 = \langle x,y \rangle, x^4=1, y^5=1; yx= xy^4.$

If c is an element of order 5 in G then a and b cannot both commute with c, since G is non-Abelian. We have the following possibilities

(1) $ac=ca, bc=c^4b$.

(2) $ac=c^4a, bc=c^4b$.

the first relation entails:

$$abc=c^4ab;$$

and the second ensure that

$$abc=c^{16}ab=c\,ab,$$

so that the two possibilities yield the same isomorphic type. Moreover, we may take one of a or b arbitrarily as the generator permuting with c. Hence we have

(iii) $G_5= \langle a,b,c\rangle,\ c^5=1,\ a^2=1,\ b^2=1,\ (ab)^2=1;\ ac=c^4a,\ bc=cb.$

$= \langle x,y\rangle,\ x^2=1,\ y^{10}=1;\ xy\,x=y^9=y^{-1},$

Where we set x=a, y=bc.

It follows that $G_5$ is the dihedral group, $D_{10}$, of order 20. We have proved the following

### 2.5.6 PROPOSITION

There are five types of groups of order 20, two are Abelian of which one is cyclic and three are non-Abelian.

### 2.5.7 SUMMARY OF DEFINING RELATIONS

(i) $G_1 = \langle a\rangle,\ a^{20}=1.$

(ii) $G_2 = \langle a,b\rangle,\ a^5=1,\ b^4=1;\ ab=ba.$

(iii) $G_3 = \langle x,y\rangle,\ x^4=1,\ y^5=1;\ xy=y^2x$

(iv) $G_4= \langle x,y\rangle,\ y^5=1,\ x^4=1;\ xy=y^4x.$

(v) $G_5 = \langle x,y\rangle,\ x^2=1,\ y^{10}=1;\ xy\,x=y^{-1}.$

Furthermore, suppose G is a non-Abelian group of order $28=2^2 \times 7$.

By Sylow's Third Theorem there are 1 or 7 Sylow 2-subgroups and only 1 Sylow 7-subgroup,

$\langle y \rangle$, $y^7=1$,

which is normal in G.

The situation where we have 1 Sylow 2-subgroup will not be considered since G will be Abelian by Another Basis Theorem for Finite Abelian Groups. We consider 2-Sylow subgroups being either cyclic or the Klein 4-group.

Suppose any Sylow 2-subgroup is cyclic, say

$K=\langle x \rangle$, $x^4=1$.

Since $\langle y \rangle$, is normal in G we must have $x^{-1}yx=y^t$

for some integer t.

Moreover, since

$x^4=1$

commutes with y, we deduce that

$y = y^{t^4}$

hence that

$t^4 \equiv 1 \pmod{7}$.

A simple computation shows that

t=1 or 6.

Since our group is non-Abelian we discard the possibility that t=1 and obtain a single isomorphic type.

(iii) $G_3=\langle x,y \rangle$; $x^4=1$, $y^7=1$; $x^{-1}yx=y^{-1}$.

For the situation where the Sylow 2-subgroup is the Klein 4-group we may write

$K = \langle a,b \rangle,\ a^2=1,\ b^2=1,\ (ab)^2 = 1.$

Then a and b cannot both commute with y since G is non-Abelian.

We have the following possibilities

(1) $ay = ya,\ by=y^6b.$

(2) $ay = y^6a,\ by=y^6b.$

The first relation shows that

$$aby = y^6ab,$$

and the second ensures that

$$aby = y^{36}ab = yab.$$

That is, the two possibilities yield the same isomorphic type. Hence we have

(iv) $G_4= \langle a,b,y \rangle,\ y^7=1, a^2=1, b^2=1;\ (ab)^2=1,\ ay=ya,\ byb=y^{-1}.$

We can write

$G_4 = \langle u,v \rangle,\ u^2=1,\ v^{14}=1;\ uvu=v^{-1},$

Where we set

$$u=b,\ v=ay.$$

In the later presentation, $G_4$ is revealed as the dihedral group, $D_{14}$ of order 28.

We have therefore proved the following

**2.5.8 PROPOSITION**

There are four classes of groups of order 28, one is cyclic, one is Abelian and two are non-Abelian.

### 2.5.9 SUMMARY OF DEFINING RELATIONS

(i) $G_1 = \langle x \rangle$ , $x^{28}=1$.

(ii) $G_2 = \langle a,b \rangle$, $a^4=1$, $b^7=1$; $ab=ba$.

(iii) $G_3 = \langle x,y \rangle$, $x^4=1$, $y^7=1$; $xyx=y^{-1}$.

(iv) $G_4 = \langle u,v \rangle$, $u^2=1$, $v^{14}$; $uvu=v^{-1}$.

Hans, Bettina and O'Brien (1999) announced a significant step in providing a solution to the group construction problem in its original form by developing practical algorithms to construct or enumerate the groups of a given order in one of their works. They enumerated the 49487365422 groups of order $2^{10}$ and determined explicitly the 423164062 remaining groups of order at most 2000. Summary of their findings is listed in the table below.

In her work, (Manalo ,2001) presented a systematic method for classifying groups of small orders. Classifying groups usually arise when trying to distinguish the number of non-isomorphic groups of order n. She started by developing a sample run of Groups 32 program which shows the orders of the elements for the group $S_3$ and $C_4$. The groups 32 package can be accessed at http://www.math.ucsd.edu/ujwavrik the orders command tells us the number of elements of each orders of the group.

Hans (2001) introduced three practical algorithms to construct certain finite groups up to isomorphism. The first one can be used to construct all soluble groups of a given order. This method can be restricted to compute soluble groups with certain properties such as nilpotent, non-nilpotent or super soluble groups. The second algorithm can be used to determine the groups of order $p^n q$ with a normal Sylow subgroup for distinct primes p and q. The third method is a general method to construct finite group used to compute insoluble groups the above mainly targets groups of prime orders which are useful in the area of determining the subnormal series. The list of their ten most difficult orders is shown in Table 7 below.

**Table 2.3:** Ten most difficult orders

| **Order** | **Number** |
|---|---|
| $2^{10}$ | 49487365422 |
| $2^9.3$ | 408641062 |
| $2^9$ | 10494213 |
| $2^8.5$ | 1116461 |
| $2^8.3$ | 1090235 |
| $2^8.7$ | 1083553 |
| $2^7.3.5$ | 241004 |
| $27.3^2$ | 157877 |
| $2^8$ | 56092 |
| $2^6$-.$3^3$ | 47937 |

Audu (1988b) found the number of transitive p-groups of degree $p^2$. Audu and Momoh presented the classification of p-groups of degree $p^3$.

Most of the work in group classification up to isomorphic looked at groups of orders that are powers of a prime. It therefore became pertinent to work at groups of orders a product of primes such as sp, spq where s,p and q are distinct primes with a view of determining their non-Abelian isomorphic types. The congruence relationship between these primes, that is for $p \equiv k \pmod{s}$ where k is an integer $1 \leq k < s$ was mainly used. This helped to determine the number of non-Abelian isomorphic types in each congruence class and the values of k that will guarantee non-existence of non-Abelian isomorphic type.

# CHAPTER THREE
# METHODS AND GENERATION OF NON-ABELIAN ISOMORPHIC TYPES

Groups factorizable into products of two primes s and p and s,p and q respectively were mainly considered. The use of the list of primes listed in Appendix 1 and the use of the conventional ways of determining the non-abelian isomorphic groups of such orders will also be made.

The scheme in Appendix II was developed to determine the numbers of integer t whose powers of s gave a remainder modulo 1 after division by p in each case.

It is written with HTML and PHP and PHP is Hyper Text Preprocessor and hosted at http://www.cenpece.org/modulo/. HTML is used because it was expected to run on a web browser which is the purpose of maximizing resources which are readily available on web browsers and can always be updated. PHP is a programming language which shares similar syntax with C++, C# and other generic languages. PHP runs seamlessly with database applications such as MySQL and Oracle Database.

It can be run on any kind of system with any form of internet connection or connection of an apache server.

The congruence modulo project can be extended to store a couple of values in the database to make it better for future usage.

Actually, when a group of order is n factorizable into two prime sp such that

$p \equiv 1 \pmod{s}$ and through the relation $t^s \equiv 1 \pmod{p}$, the scheme gives all the possible values of r in the interval $1 < t < p$. We will, however, not only outline different values of t but will also put up defining relations of such non-Abelian isomorphic types that would be obtained from different values of r.

This was also done for cases where $p \equiv k \pmod{s}$ for $k > 1$.

### 3.0 NON ABELIAN ISOMORPHIC TYPES OF SOME GROUPS OF ORDER 2p.

Here the non-Abelian isomorphic types of some groups of order 2p, where (2, p) = 1 and $p \equiv 1 \pmod 2$ were obtained. Actually primes numbers not equal to 2 are congruent to1 modulo 2. We shall be using elements a and b as generators of the groups until otherwise stated.

To obtain the non-Abelian group of order 6, we first observe that 6 = 2 x 3 and that a group G of order 6, can be isomorphic to direct product of two cyclic groups of orders 3 and 2. Hence $G = \{e, a, a^2, b, ba, ba^2\}$, where $a \in C_3$ such that $a^3 = e$ and $b \in C_2$ such that $b^2 = e$

Since $b \notin C_3$ and to obtain closure for the elements of G, we see that ab = ba or $ba^2$, but ab = ba will be ruled out since our interest is on the non-Abelian isomorphic type of G.

Therefore, for $ab = ba^2$ we have

$$(ab)^2 = ab\,ab = abba^2 = aea^2 = a^3 = e.$$

Hence $G \cong \langle a \rangle \times \langle b \rangle$ such that

$$a^3 = b^2 = e \text{ and } ab = ba^2.$$

For the group G of order 10 we follow similar steps as above to see that G can be of the direct products of cyclic groups $C_5$ and $C_2$ of orders 5 and 2 respectively.

Since if $a^5 = e = b^2$ then $ab = ba^2$ or $ba^3$ cannot satisfy closure property. That is if $ab = ba^2$ then

$$(ab)^2 = abab = ba^2ab = ba^3b$$

$$(ab)^3 = (ab)^2ab = ba^3bab = ba^3bba^2 = ba^5 = b$$

$$(ab)^4 = (ab)^3 = bab = bba^2 = a^2$$

$$(ab)^5 = (ab)^4ab = a^2ab = a^3b$$

and none gave the identity element.

Also for $ab = ba^3$, $(ab)^2$, $(ab)^3$ , $(ab)^4$ and $(ab)^5$ cannot give the identity

For $a = ba^4$ , we have

$$(ab)^2 = abab = ba^4ab = b^2 = e, \text{ the identity.}$$

Hence $G = \langle a, b \rangle \cong C_5 \times C_2 = \langle a \rangle \times \langle b \rangle$.

This is a non-Abelian isomorphic type of a group G of order 10.

For a group G of order $14 = 2 \times 7$, we see that $G \cong C_7 \times C_2$

But $C_7 = \{e, a, a^2, a^3, a^4, a^5, a^6\}$ and

$$C_2 = \{e, b\} \text{ with } a^5 = b^2 = e.$$

Hence $G = \{e, a, a^2, a^3, a^4, a^5, a^6, b, a, a^2b, a^3b, a^4b, a^5b, a^6b\}$.

Since $b \notin C_7$ which would have made it to have order different from 2, we show that $ab = ba^2, ba^3, ba^4, ba^5$ or $ba^6$. Close scrutiny shows that

$$ab = ba^6 \text{ and } (ab)^2 = abab = ba^6ab = b^2 = e.$$

Hence $G = \langle a, b \rangle \cong \langle a \rangle \times \langle b \rangle$

and $a^7 = e = b^2$ , with $ab = ba^6$ and $(ab)^2 = e$.

This gave a non-Abeian isomorphic type.

For a group G of order $22 = 11 \times 2$ we see that $G = \langle a , b \rangle \cong C_{11} \times C_2$

with $a^{11} = b^2 = e$ , $ab = ba^{10}$ and $(ab)^2 = ab\,ab = ba^{10}ab = b^2 =e$.

We also observed that for any group of order 6, 14, or 22… that

$2^2 \equiv 1 \pmod 3$, $\quad 6^2 \equiv 1 \pmod 7$ or $10^2 \equiv 1 \pmod{11}$

indicating that from

$1< t < 7$ or $1 < t < 11$

and that t took the value $p – 1$ in each case.

Also, $5 \equiv 1 \pmod 2$,

$7 \equiv 1 \pmod 2$, and

$11 \equiv 1 \pmod 2$.

Our scheme showed that for any group of order n = 2p, where p is a prime, has only one value for t and this value is always p – 1 for distinct values of p.

## 3.1 NON-ABELIAN GROUPS OF ORDER n = 3p with 100 < p < 2000 and p ≡ 1(mod 3)

For groups of order n = 3p, our scheme gave the following results:

If we take a and b to be elements of order 3 and p respectively,

i.e. $a^3 = b^p = 1$, we have the following non-Abelian isomorphic types for each p:

For a group of order 21 = 3 x 7, we see that $G = \langle a, b \rangle \cong C_7 \times C_3$

with $b^7 = a^3 = e$ , $ba = ab^2$ and $(ba)^3 = e$.

This is a non-Abelian isomorphic type.

Hence for a group of order 21 that $2^3 \equiv 1 \pmod 7$. Here again t is within the range

$1 < t < 7$.

For any group G of order 39 = 3 x 13, we have that

$$G = C_{13} \times C_3 \cong \langle a \rangle \times \langle b \rangle$$

with $b^{13} = a^3 = e$ and $ba = ab^3$

For closure we have

$$(ba)^2 = ab^3ab^3 = ab^2ab^6 = abab^9 = aab^{12} = a^2b^{12}$$

$$(ba)^3 = baa^2b^{12} = e$$

Again for $ba = ab^9$, we have

$$(ba)^2 = ab^9ab^9 = ab^8ab^5 = ab^7ab = ab^6ab^{10}$$

$$= ab^5ab^6 = ab^4ab^2 = ab^3ab^{11} = ab^2ab^7$$

$$= abab^3 = a^2b^{12}$$

$$(ba)^3 = baa^2b^{12} = e$$

But 9 is a power of 3 and the first case stands.

For any group G of order 57 = 3 x 19, we have that

$G = C_{19} \times C_3 \cong \langle a \rangle \times \langle b \rangle$

with $b^{19} = a^3 = e$ and (i) $ba = ab^7$ (ii) $ba = ab^{11}$ where closure properties are as follows:

(i) $(ba)^2 = ab^7ab^7 = ab^6ab^{14} = ab^5ab^2 = ab^4ab^9 = ab^3ab^{16} = ab^2ab^4 = abab^{11}$

$= a^2b^{18}$.

$(ba)^3 = baa^2b^{18} = e$

(ii) $ba = ab^{11}$ ;

$(ba)^2 = ab^{11}ab^{11} = ab^{10}ab^3 = ab^9ab^{14} = ab^8ab^6$

$= ab^7ab^{17} = ab^6ab^9 = ab^5ab = ab^4ab^{12} = ab^3ab^4$

$= ab^2ab^{15} = abab^7 = a^2b^{18}$.

$(ba)^3 = baa^2b^{18} = e$

This shows that G is isomorphic as follows:

(i) $G \cong \langle a \rangle \times \langle b \rangle$

with $b^{19} = a^3 = e$ and $ba = ab^7$ , and

(ii) $G \cong \langle a \rangle \times \langle b \rangle$

with $b^{19} = a^3 = e$ and $ba = ab^{11}$.

1. For subgroups of orders (3)(109) we have

(i) $G_1 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1}ba = b^{45}$

(ii) $G_2 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1}ba = b^{63}$

2. For subgroups of orders (3)(139) we have

(i) $G_1 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1}ba = b^{42}$

(ii) $G_2 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1} ba = b^{96}$

3. For subgroups of orders (3)(199) we have

(i) $G_1 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1} ba = b^{92}$

(ii) $G_2 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1} ba = b^{106}$

4. For subgroups of orders (3)(229) we have

(i) $G_1 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1}ba = b^{94}$

3. For subgroups of orders (3)(409) we have

(i) $G_1 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1} ba = b^{53}$

(ii) $G_2 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1} ba = b^{355}$

4. For subgroups of orders (3)(439) we have

(i) $G_1 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1} ba = b^{171}$

(ii) $G_2 \langle a \rangle \times \langle b \rangle$; where $a^{-1} ba = b^{267}$

5. For subgroups of orders (3)(619) we have

(i) $G_1 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1} ba = b^{252}$

(ii) $G_2 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1} ba = b^{366}$

6. For subgroups of orders (3)(739) we have

(i) $G_1 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1} ba = b^{320}$

(ii) $G_2 \cong \langle a \rangle \times \langle b \rangle$, $a^{-1} ba = b^{418}$

7. For subgroups of orders (3)(829) we have

(i) $G_1 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1} ba = b^{125}$

(ii) $G_2 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1} ba = b^{703}$

8. For subgroups of orders (3)(919) we have

(i) $G_1 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1}ba = b^{52}$

(ii) $G_2 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1}ba = b^{866}$

9. For subgroups of orders (3)(1009) we have

(i) $G_1 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1}ba = b^{374}$

(ii) $G_2 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1}ba = b^{634}$

10. For subgroups of orders (3)(1129) we have

(i) $G_1 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1}ba = b^{387}$

(ii) $G_2 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1}ba = b^{741}$

11. For subgroups of orders (3)(1279) we have

(i) $G_1 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1}ba = b^{504}$

(ii) $G_2 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1}ba = b^{774}$

12. For subgroups of orders (3)(1459) we have

(i) $G_1 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1}ba = b^{339}$

(ii) $G_2 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1}ba = b^{1119}$

13. For subgroups of orders (3)(1579) we have

(i) $G_1 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1}ba = b^{639}$

(ii) $G_2 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1}ba = b^{939}$

14. For subgroups of orders (3)(1699) we have

(i) $G_1 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1}ba = b^{397}$

(ii) $G_2 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1}ba = b^{1301}$

15. For subgroups of orders (3)(1999) we have

(i) $G_1 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1} ba = b^{808}$

(ii) $G_2 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1} ba = b^{1190}$

16. For subgroups of orders (3)(127) we have

(i) $G_1 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1} ba = b^{19}$

(ii) $G_2 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1} ba = b^{107}$

17. For subgroups of orders (3)(307) we have

(i) $G_1 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1} ba = b^{17}$

(ii) $G_2 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1} ba = b^{289}$.

18. For subgroups of orders (3)(457) we have

(i) $G_1 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1} ba = b^{133}$

(ii) $G_2 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1} ba = b^{323}$

19. For subgroups of orders (3)(577) we have

(i) $G_1 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1} ba = b^{213}$

(ii) $G_2 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1} ba = b^{363}$

20. For subgroups of orders (3)(757) we have

(i) $G_1 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1} ba = b^{27}$

(ii) $G_2 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1} ba = b^{729}$

21. For subgroups of orders (3)(907) we have

(i) $G_1 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1} ba = b^{384}$

(ii) $G_2 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1} ba = b^{522}$

22. For subgroups of orders (3)(1117) we have

(i) $G_1 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1} ba = b^{120}$

(ii) $G_2 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1} ba = b^{996}$

23. For subgroups of orders (3)(1237) we have

(i) $G_1 \cong g\langle a \rangle \times \langle b \rangle$; where $a^{-1} ba = b^{300}$

(ii) $G_2 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1} ba = b^{936}$

24. For subgroups of orders (3)(1597) we have

(i) $G_1 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1} ba = b^{222}$

(ii) $G_2 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1} ba = b^{1374}$

25. For subgroups of orders (3)(1747) we have

(i) $G_1 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1} ba = b^{371}$

26. For subgroups of orders (3)(1987) we have

(i) $G_1 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1} ba = b^{647}$

(ii) $G_2 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1} ba = b^{1339}$

27. For subgroups of orders (3)(103) we have

(i) $G_1 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1} ba = b^{46}$

(ii) $G_2 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1} ba = b^{56}$

28. For subgroups of orders (3)(223) we have

(i) $G_1 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1} ba = b^{31}$

(ii) $G_2 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1} ba = b^{183}$

29. For subgroups of orders (3)(433) we have

(i) $G_1 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1} ba = b^{198}$

(ii) $G_2 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1} ba = b^{234}$

30. For subgroups of orders (3)(643) we have

(i) $G_1 \cong \langle a\rangle \times \langle b\rangle$; where $a^{-1}ba = b^{177}$

(ii) $G_2 \cong \langle a\rangle \times \langle b\rangle$; where $a^{-1}ba = b^{465}$

31. For subgroups of orders (3)(883) we have

(i) $G_1 \cong \langle a\rangle \times \langle b\rangle$; where $a^{-1}ba = b^{337}$

(ii) $G_2 \cong \langle a\rangle \times \langle b\rangle$; where $a^{-1}ba = b^{545}$

32. For subgroups of orders (3)(1093) we have

(i) $G_1 \cong \langle a\rangle \times \langle b\rangle$; where $a^{-1}ba = b^{151}$

(ii) $G_2 \cong \langle a\rangle \times \langle b\rangle$; where $a^{-1}ba = b^{941}$

33. For subgroups of orders (3)(1123) we have

(i) $G_1 \cong \langle a\rangle \times \langle b\rangle$; where $a^{-1}ba = b^{33}$

(ii) $G_2 \cong \langle a\rangle \times \langle b\rangle$; where $a^{-1}ba = b^{1089}$

34. For subgroups of orders (3)(1303) we have

(i) $G_1 \cong \langle a\rangle \times \langle b\rangle$; where $a^{-1}ba = b^{95}$

(ii) $G_2 \cong \langle a\rangle \times \langle b\rangle$; where $a^{-1}ba = b^{1207}$

35. For subgroups of orders (3)(1453) we have

(i) $G_1 \cong \langle a\rangle \times \langle b\rangle$; where $a^{-1}ba = b^{693}$

(ii) $G_2 \cong \langle a\rangle \times \langle b\rangle$; where $a^{-1}ba = b^{759}$

36. For subgroups of orders (3)(14833) we have

(i) $G_1 \cong \langle a\rangle \times \langle b\rangle$; where $a^{-1}ba = b^{38}$

(ii) $G_2 \cong \langle a\rangle \times \langle b\rangle$; where $a^{-1}ba = b^{1444}$

37. For subgroups of orders (3)(1693) we have

(i) $G_1 \cong \langle a\rangle \times \langle b\rangle$; where $a^{-1}ba = b^{433}$

(ii) $G_2 \cong \langle a\rangle \times \langle b\rangle$; where $a^{-1}ba = b^{1259}$

38. For subgroups of orders (3)(1783) we have

(i) $G_1 \cong \langle a\rangle \times \langle b\rangle$; where $a^{-1}ba = b^{193}$

(ii) $G_2 \cong \langle a\rangle \times \langle b\rangle$; where $a^{-1}ba = b^{1589}$

39. For subgroups of orders (3)(1993) we have

(i) $G_1 \cong \langle a\rangle \times \langle b\rangle$; where $a^{-1}ba = b^{312}$

(ii) $G_2 \cong \langle a\rangle \times \langle b\rangle$; where $a^{-1}ba = b^{1680}$

We summarize the above findings in as follows:

### 3.2 LEMMA

If $100 < p < 2000$ and $p \equiv 1 \pmod 3$ then groups of order $n = 3p$ have at most two non - Abelian isomorphic types.

**PROOF:** This follows from the examples generated above. For a group of order $n = 3p$, $p \equiv 1 \pmod 3$ there are only two values of t such that $t^3 \equiv 1 \pmod p$, $t_1$ and $t_2$, say. Any other value for $t \neq t_1$ or $t_2$ must be a must a power of one of the $t_1$ or $t_2$ . Hence such group has two non-abelian isomorphic types

### 3.3 FOR SUBGROUPS OF ORDER 3p WHERE 2000 < p < 4000

Further application of our scheme on groups of order $n = 3p$, for distinct primes, p are as follows:

For each prime p the following non-Abelian types, together with their defining relations are displayed (where a and b are two generators such that $a^3 = b^p = 1$):

1. For subgroups of order (3)(2011) we have

(i) $G_1 \cong \langle a\rangle \times \langle b\rangle$; where $a^{-1}ba = b^{205}$

(ii) $G_2 \cong \langle a\rangle \times \langle b\rangle$; where $a^{-1}ba = b^{1805}$

2. For subgroups of order (3)(2131) we have

(i) $G_1 \cong \langle a\rangle \times \langle b\rangle$; where $a^{-1}ba = b^{468}$

(ii) $G_2 \cong \langle a\rangle \times \langle b\rangle$; where $a^{-1}ba = b^{1662}$

3. For subgroups of order (3)(2251) we have

(a) $G_1 \cong \langle a\rangle \times \langle b\rangle$; where $a^{-1}ba = b^{708}$

(b) $G_2 \cong \langle a\rangle \times \langle b\rangle$; where $a^{-1}ba = b^{1542}$

4. For subgroups of order (3)(2311) we have

(i) $G_1 \cong \langle a\rangle \times \langle b\rangle$; where $a^{-1}ba = b^{882}$

(ii) $G_2 \cong \langle a\rangle \times \langle b\rangle$; where $a^{-1}ba = b^{1428}$

5. For subgroups of order (3)(2371) There are:

(i) $G_1 \cong \langle a\rangle \times \langle b\rangle$; where $a^{-1}ba = b^{464}$

(ii) $G_2 \cong \langle a\rangle \times \langle b\rangle$; where $a^{-1}ba = b^{1906}$

6. For subgroups of order (3)(2671) we have

(i) $G_1 \cong \langle a\rangle \times \langle b\rangle$; where $a^{-1}ba = b^{544}$

(ii) $G_2 \cong \langle a\rangle \times \langle b\rangle$; where $a^{-1}ba = b^{2126}$

7. For subgroups of order (3)(2971) we have

(i) $G_1 \cong \langle a\rangle \times \langle b\rangle$; where $a^{-1}ba = b^{54}$

(ii) $G_2 \cong \langle a\rangle \times \langle b\rangle$; where $a^{-1}ba = b^{2916}$

8. For subgroups of order (3)(3001) we have

(i) $G_1 \cong \langle a\rangle \times \langle b\rangle$; where $a^{-1}ba = b^{934}$

(ii) $G_2 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1}ba = b^{2066}$

9. For subgroups of order (3)(3181) we have

(i) $G_1 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1}ba = b^{440}$

(ii) $G_2 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1}ba = b^{2740}$

10. For subgroups of order (3)(3331) we have

(i) $G_1 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1}ba = b^{1463}$

(ii) $G_2 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1}ba = b^{1867}$

11. For subgroups of order (3)(3511) we have

(i) $G_1 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1}ba = b^{59}$

(ii) $G_2 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1}ba = b^{3481}$

12. For subgroups of order (3)(3691) we have

(i) $G_1 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1}ba = b^{474}$

(ii) $G_2 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1}ba = b^{3216}$

13. For subgroups of order (3)(3931) we have

(i) $G_1 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1}ba = b^{617}$

(ii) $G_2 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1}ba = b^{3313}$

## 3.4 LEMMA

For groups of order $n = 3p$, where $2000 < p < 4000$, $p \equiv 1 \pmod 3$ there can be a only two non-abelian isomorphic type.

Proof: This is just what we proved in Lemma 3.2.

## 3.5 GROUPS OF ORDER 5p WHERE p ≡ 1(mod 5) AND 100 < p < 2000.

In this case the following situation occur for $a^5 = b^p = 1$, where a and b are generators of order 5 and p respectively.

1. For subgroups of order (5)( 131), we have

   i. $G_1 \cong \langle a \rangle \times \langle b \rangle$, where $a^{-1}ba = b^{53}$

   ii. $G_2 \cong \langle a \rangle \times \langle b \rangle$ where $a^{-1}ba = b^{58}$

   iii. $G_3 \cong \langle a \rangle \times \langle b \rangle$ where $a^{-1}ba = b^{61}$

   iv. $G_4 \cong \langle a \rangle \times \langle b \rangle$ where $a^{-1}ba = b^{89}$

2. For subgroup of order (5)(251) we have

   i. $G_1 \cong \langle a \rangle \times \langle b \rangle$, where $a^{-1}ba = b^{20}$

   ii. $G_2 \cong \langle a \rangle \times \langle b \rangle$ where $a^{-1}ba = b^{113}$

   iii. $G_3 \cong \langle a \rangle \times \langle b \rangle$ where $a^{-1}ba = b^{149}$

   iv. $G_4 \cong \langle a \rangle \times \langle b \rangle$ where $a^{-1}ba = b^{129}$

3. For subgroups of order (5)(251) we have

   i. $G_1 \cong \langle a \rangle \times \langle b \rangle$, where $a^{-1}ba = b^{86}$

   ii. $G_2 \cong \langle a \rangle \times \langle b \rangle$ where $a^{-1}ba = b^{90}$

   iii. $G_3 \cong \langle a \rangle \times \langle b \rangle$ where $a^{-1}ba = b^{153}$

   iv. $G_4 \cong \langle a \rangle \times \langle b \rangle$ where $a^{-1}ba = b^{232}$

4. For subgroups of order (5)(461) we have

   i. $G_1 \cong \langle a \rangle \times \langle b \rangle$, where $a^{-1}ba = b^{88}$

   ii. $G_2 \cong \langle a \rangle \times \langle b \rangle$ where $a^{-1}ba = b^{114}$

   iii. $G_3 \cong \langle a \rangle \times \langle b \rangle$ where $a^{-1}ba = b^{351}$

   iv. $G_4 \cong \langle a \rangle \times \langle b \rangle$ where $a^{-1}ba = b^{368}$

5. For subgroups of order (5)(491) we have

   i. $G_1 \cong \langle a\rangle \times \langle b\rangle$, where $a^{-1}ba = b^{101}$

   ii. $G_2 \cong \langle a\rangle \times \langle b\rangle$ where $a^{-1}ba = b^{183}$

   iii. $G_3 \cong \langle a\rangle \times \langle b\rangle$ where $a^{-1}ba = b^{316}$

   iv. $G_4 \cong \langle a\rangle \times \langle b\rangle$ where $a^{-1}ba = b^{381}$

6. For subgroup of order (5)(641) we have

   i. $G_1 \cong \langle a\rangle \times \langle b\rangle$, where $a^{-1}ba = b^{357}$

   ii. $G_2 \cong \langle a\rangle \times \langle b\rangle$ where $a^{-1}ba = b^{47}$

   iii. $G_3 \cong \langle a\rangle \times \langle b\rangle$ where $a^{-1} ba = b^{531}$

   iv. $G_4 \cong \langle a\rangle \times \langle b\rangle$ where $a^{-1} ba = b^{562}$

7. For subgroups of order (5)(881) we have

   i. $G_1 \cong \langle a\rangle \times \langle b\rangle$, where $a^{-1}ba = b^{268}$

   ii. $G_2 \cong \langle a\rangle \times \langle b\rangle$, where $a^{-1}ba = b^{286}$

   iii. $G_3 \cong \langle a\rangle \times \langle b\rangle$, where $a^{-1}ba = b^{463}$

   iv. $G_4 \cong \langle a\rangle \times \langle b\rangle$, where $a^{-1}ba = b^{744}$

8. For subgroup of order (5)(941) we have

   i. $G_1 \cong \langle a\rangle \times \langle b\rangle$, where $a^{-1}ba = b^{349}$

   ii. $G_2 \cong \langle a\rangle \times \langle b\rangle$ gp{a} x gp{b}, where $a^{-1}ba = b^{364}$

   iii. $G_3 \cong \langle a\rangle \times \langle b\rangle$, where $a^{-1}ba = b^{412}$

   iv. $G_4 \cong \langle a\rangle \times \langle b\rangle$, where $a^{-1}ba = b^{756}$

9. For subgroups of order (5)(1061) we have

   i. $G_1 \cong \langle a\rangle \times \langle b\rangle$, where $a^{-1}ba = b^{220}$

ii. $G_2 \cong \langle a \rangle \times \langle b \rangle$, where $a^{-1}ba = b^{381}$

iii. $G_3 \cong \langle a \rangle \times \langle b \rangle$, where $a^{-1}ba = b^{655}$

iv. $G_4 \cong \langle a \rangle \times \langle b \rangle$, where $a^{-1}ba = b^{862}$

10. For subgroups of order (5)(1301) we have

i. $G_1 \cong \langle a \rangle \times \langle b \rangle$, where $a^{-1}ba = b^{163}$

ii. $G_2 \cong \langle a \rangle \times \langle b \rangle$, where $a^{-1}ba = b^{549}$

iii. $G_3 \cong \langle a \rangle \times \langle b \rangle$, where $a^{-1}ba = b^{870}$

iv. $G_4 \cong \langle a \rangle \times \langle b \rangle$, where $a^{-1}ba = b^{1019}$

11. For subgroups of order (5)(1511) we have

i. $G_1 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1}ba = b^{534}$

ii. $G_2 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1}ba = b^{631}$

iii. $G_3 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1}ba = b^{768}$

iv. $G_4 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1}ba = b^{1088}$

12. For subgroups of order (5)(1811) we have

i. $G_1 \cong \langle a \rangle \times \langle b \rangle$, where $a^{-1}ba = b^{433}$

ii. $G_2 \cong \langle a \rangle \times \langle b \rangle$, where $a^{-1}ba = b^{956}$

iii. $G_3 \cong \langle a \rangle \times \langle b \rangle$, where $a^{-1}ba = b^{1040}$

iv. $G_4 \cong \langle a \rangle \times \langle b \rangle$, where $a^{-1}ba = b^{1192}$

13. For subgroups of order (5)(1931) we have

i. $G_1 \cong \langle a \rangle \times \langle b \rangle$, where $a^{-1}ba = b^{1101}$

ii. $G_2 \cong \langle a \rangle \times \langle b \rangle$, where $a^{-1}ba = b^{1410}$

iii. $G_3 \cong \langle a \rangle \times \langle b \rangle$, where $a^{-1}ba = b^{1467}$

### 3.6 LEMMA

If $100 < p < 2000$, $p \equiv 1 \pmod 5$, there are at most four non-Abelian Isomorphic types of groups of order 5p.

**PROOF:** This follows from the examples generated above. A group of order $n = 5p$, $p \equiv 1 \pmod 5$ has only four values of t such that $t^5 \equiv 1 \pmod p$, $t_1$, $t_2$, $t_3$ and $t_4$, say. Any other value for $t \neq t_1, t_2, t_3$, or $t_4$ must be a must a power of any one of them. Hence such group has at most four non-abelian isomorphic types

### 3.7 FOR GROUPS OF ORDER n = 5p, FOR 2000 < p < 4000.

Here we also assume two element generators a and b such that

$a^5 = b^p = 1$ and $p \equiv 1 \pmod 5$. The following non-Abelian types are obtained:

1. For subgroups of order (5)(2011) we have

(i) $G_1 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1}ba = b^{798}$

(ii) $G_2 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1}ba = b^{1328}$

(iii) $G_3 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1}ba = b^{1948}$

(iv) $G_4 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1}ba = b^{1958}$

2. For subgroups of order (5)(2131) we have

(i) $G_1 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1}ba = b^{832}$

(ii) $G_2 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1}ba = b^{1734}$

(iii) $G_3 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1}ba = b^{1780}$

(iv) $G_4 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1}ba = b^{2046}$

3. For subgroups of order (5)(2251) we have

(i) $G_1 \cong \langle a\rangle \times \langle b\rangle$; where $a^{-1}ba = b^{361}$

(ii) $G_2 \cong \langle a\rangle \times \langle b\rangle$; where $a^{-1}ba = b^{2014}$

(iii) $G_3 \cong \langle a\rangle \times \langle b\rangle$; where $a^{-1}ba = b^{2232}$

4. For subgroups of order (5)(2341) we have

(i) $G_1 \cong \langle a\rangle \times \langle b\rangle$; where $a^{-1}ba = b^{735}$

(ii) $G_2 \cong \langle a\rangle \times \langle b\rangle$; where $a^{-1}ba = b^{809}$

(iii) $G_3 \cong \langle a\rangle \times \langle b\rangle$; where $a^{-1}ba = b^{1342}$

5. For subgroups of order (5)(2521) we have

(i) $G_1 \cong \langle a\rangle \times \langle b\rangle$; where $a^{-1}ba = b^{757}$

(ii) $G_2 \cong \langle a\rangle \times \langle b\rangle$; where $a^{-1}ba = b^{782}$

(iii) $G_3 \cong \langle a\rangle \times \langle b\rangle$; where $a^{-1}ba = b^{1442}$

(iv) $G_4 \cong \langle a\rangle \times \langle b\rangle$; where $a^{-1}ba = b^{2060}$

6. For subgroups of order (5)(2731) we have

(i) $G_1 \cong \langle a\rangle \times \langle b\rangle$; where $a^{-1}ba = b^{742}$

(ii) $G_2 \cong \langle a\rangle \times \langle b\rangle$; where $a^{-1}ba = b^{1233}$

7. For subgroups of order (5)(2851) we have

(i) $G_1 \cong \langle a\rangle \times \langle b\rangle$; where $a^{-1}ba = b^{45}$

(ii) $G_2 \cong \langle a\rangle \times \langle b\rangle$; where $a^{-1}ba = b^{887}$

(iii) $G_3 \cong \langle a\rangle \times \langle b\rangle$; where $a^{-1}ba = b^{2744}$

8. For subgroups of order (5)(3121) we have

(i) $G_1 \cong \langle a\rangle \times \langle b\rangle$; where $a^{-1}ba = b^{190}$

(ii) $G_2 \cong \langle a\rangle \times \langle b\rangle$; where $a^{-1}ba = b^{2545}$

(iii) $G_3 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1}ba = b^{3081}$

9. For subgroups of order (5)(3181) we have

(i) $G_1 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1}ba = b^{425}$

(ii) $G_2 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1}ba = b^{1714}$

10. For subgroups of order (5)(3301) we have

(i) $G_1 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1}ba = b^{454}$

(ii) $G_2 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1}ba = b^{1454}$

(iii) $G_3 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1}ba = b^{1476}$

(iv) $G_4 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1}ba = b^{3217}$

11. For subgroups of order (5)(3391) we have

(i) $G_1 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1}ba = b^{926}$

(ii) $G_2 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1}ba = b^{1805}$

(iii) $G_3 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1}ba = b^{2049}$

(iv) $G_4 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1}ba = b^{2944}$

12. For subgroups of order (5)(3931) we have

(i) $G_1 \cong \langle a \rangle \times \langle b \rangle$; where $a^{-1}ba = b^{1547}$.

## 3.8 LEMMA

For groups of order $n = 5p$, where $2000 < p < 4000$, $p \equiv 1 \pmod 5$ there can be only four non-abelian isomorphic type.

**PROOF:** From our examples above, this is just the proof of Lemma 3.6 above.

## 3.9 FOR GROUPS OF ORDER n = 7p SUCH THAT p ≡ 1 (mod 7) AND 20 < p < 2000.

For any group G of order 203 = 7 x 29 we will have

$G \cong \langle a \rangle \times \langle b \rangle$;

with $b^{29} = a^7 = e$ and for different values of in the defining relation $ba = ab^t$ we obtain the following:

(i) $ba = ab^7$,

(ii) $ba = ab^{16}$ ,

(iii) $ba = ab^{20}$,

(iv) $ba = ab^{23}$,

(v) $ba = ab^{24}$ and

(vi) $ab = ab^{25}$

For clarity, we show the closure properties of (i) and (ii) as follows:

with $ba = ab^7$,

$(ba)^2 = ab^7ab^7 = ab^6ab^{14} = ab^5ab^{21} = ab^4ab^{28} = ab^3ab^6 = ab^2ab^{13} = abab^{20} = a^2b^{27}$

$(ba)^3 = a^2b^{27}ab^7 = \ldots = a^3b^{22}$

$(ba)^4 = a^3b^{22}ab^7 = \ldots = a^4b^{16}$

$(ba)^5 = a^4b^{16}ab^7 = a^4b^{15}ab^{14} = \ldots = a^5b^3$

$(ba)^6 = a^5b^3ab^7 = \ldots = a^6b^{28}$

$(ba)^7 = baa^6b^{28} = e$

With $ba = ab^{16}$ we have

$(ba)^2 = ab^{16}ab^{16} = \ldots = a^2b^{11}$

$(ba)^3 = a^2b^{11}ab^{16} = \ldots = a^3b^{18}$

$(ba)^4 = a^3b^{18}ab^{16} = \ldots = a^4b^{14}$

$(ba)^5 = a^4b^{14}ab^{16} = \ldots = a^5b^8$

$(ba)^6 = a^5b^8ab^{16} = \ldots = a^6b^{28}$

$(ba)^7 = baa^6b^{28} = e.$

With $ba = ab^{20}$

$(ba)^2 = ab^{20}ab^{20} = ab^{19}ab^{11} = \ldots = a^2b^{14}$

$(ba)^3 = a^2b^{14}ab^{20} = \ldots = a^3b^{10}$

$(ba)^4 = a^3b^{10}ab^{20} = a^3b^9ab^{11} = \ldots = a^4b^{17}$

$(ba)^5 = a^4b^{17}ab^{20} = \ldots = a^5b^{12}$

$(ba)^6 = a^5b^{12}ab^{20} = a^5b^{12}ab^{20} = a^5b^{11}ab^{11} \ldots = a^6b^{28}$

$(ba)^7 = baa^6b^{28} = e.$

With $ba = ab^{23}$, we have

$(ba)^2 = ab^{23}ab^{23} = ab^{22}ab^{17} = \ldots = a^2b$

$(ba)^3 = a^2bab^{23} = a^3b^{17}$

$(ba)^4 = a^3b^{17}ab^{23} = a^3b^{16}ab^{17} = a^3b^{15}ab^{11} = \ldots = a^4b^8$

$(ba)^5 = a^4b^8ab^{23} = a^4b^7ab^{17} = \ldots = a^5b^4$

$(ba)^6 = a^5b^4ab^{23} = a^5b^3ab^{17} = a^5b^2ab^{11} = \ldots = a^6b^{28}$

$\therefore (ba)^7 = baa^6b^{28} = e.$

Similarly for a group of order 21 that $2^3 \equiv 1 \pmod 7$. Here again r is within the range $1 < t < 7$.

With $ba = ab^{24}$, similar approach shows that

$ba = ab^{24}$ is of order 7.

Here we make use of the fact that each subgroup is a two element generator, a and b say, with $a^7 = b^p = 1$.

1. For p = 29 and for a subgroup of order (7) (29) we have the following:

$G_1 \cong \langle a \rangle \times \langle b \rangle$; with $a^{-1}ba = b^t$

Where t = 7, 16, 23, 24, 25

It is easily verified that each of the elements $ba = ab^7$, $ba = ab^{16}$, $ba = ab^{20}$, $ba = ab^{23}$, $ba = ab^{24}$, and $ba = ab^{25}$ have order 7 in their respective non-Abelian groups. That is to say that the elements $ba = ab^{t_1}, ba = ab^{t_2}, \ldots,$ form different non-Abelian groups of order sp have order s respectively.

2. For p = 43 and for a subgroup of (7) (43) we have the following

$G_1 \cong \langle a \rangle \times \langle b \rangle$; with $a^{-1}ba = b^t$

Where t = 4, 11, 21, 35, 41

3. For p = 71 and for a subgroup of order (7) (71) we have

$G_1 \cong \langle a \rangle \times \langle b \rangle$; with $a^{-1}ba = b^t$

Where t =30, 32, 37, 45, 48

4. For p = 113 and for subgroup of order (7) (113) have:

$G_1 \cong \langle a \rangle \times \langle b \rangle$; with $a^{-1}ba = b^t$

Where t =16, 28, 30, 49, 106, 109

6. For p = 127 and for subgroup of order (7) (127) we have.

$G_1 \cong \langle a \rangle \times \langle b \rangle$; with $a^{-1}ba = b^2$ or $b^4$ or $b^8$ or $b^{64}$

Any of the options generate the same group since $4 = 2^2$ and $8 = 2^3$, $32 = 2^5$, $64 = 2^6$

6. For p = 197 and for subgroup of order (7) (197) we have;

$G_1 \cong \langle a \rangle \times \langle b \rangle$; with $a^{-1}ba = b^t$

where t = 36, 104, 114, 164, 178

7. For p = 211 and for subgroups of order (7) (211) and for $a^7 = b^{211} = 1$ we have;

$G_1 \cong \langle a \rangle \times \langle b \rangle$; with $a^{-1}ba = b^t$

where t = 58, 123, 144, 148, 171

8. For p = 239 and for subgroups of order (7) (239) we have

$G_1 \cong \langle a \rangle \times \langle b \rangle$; with $a^{-1}ba = b^t$

where t = 10, 24, 44, 98

9. For p = 281 and for subgroup of order (7) (281) and for $a^7 = b^{281} = 1$

$G_1 \cong \langle a \rangle \times \langle b \rangle$; with $a^{-1}ba = b^t$

Where t = 59, 79, 109, 165, 181

10. For p = 449 and for a subgroup of order (7) (449) we have;

$G_1 \cong \langle a \rangle \times \langle b \rangle$; with $a^{-1}ba = b^t$

Where t = 18, 176, 285, 444

It can be observed that $t_1 = 18$ and $t_4 = 324 = 18^2$.

11. For p = 463 and for a subgroup of order (7) (463) we have

$G_1 \cong \langle a \rangle \times \langle b \rangle$; with $a^{-1}ba = b^t$

Where t = 34, 118, 230, 286, 308, 312

12. For p = 547 and for a subgroup of order (7) (547) we have;

$G_1 \cong \langle a \rangle \times \langle b \rangle$; with $a^{-1}ba = b^t$

Where t = 9, 182, 304, 520, 533, 544

13. For p = 617 and for a subgroup of order (7) (617) we have;

$G_1 \cong \langle a \rangle \times \langle b \rangle$; with $a^{-1}ba = b^t$

where t = 142, 408, 420

14. For p = 701 for subgroups of order (7) (701), we have;

$G_1 \cong \langle a \rangle \times \langle b \rangle$; with $a^{-1}ba = b^t$

where t = 19, 167, 636

15. For p = 743 for subgroup of order (7) (743), we have;

$G_1 \cong \langle a \rangle \times \langle b \rangle$; with $a^{-1} ba = b^t$

where t = 111, 328, 450, 590

16. For p = 757 and for subgroup of order (7) (757) we have;

$G_1 \cong \langle a \rangle \times \langle b \rangle$; with $a^{-1} ba = b^t$

where t = 59, 62, 77, 232, 559

17. For p = 953 and for a subgroup of order (7) (953), we have;

$G_1 \cong \langle a \rangle \times \langle b \rangle$; with $a^{-1} ba = b^t$

where t = 508, 528, 822, 559

18. For p = 967 and for a subgroup order (7) (967), we have;

$G_1 \cong \langle a \rangle \times \langle b \rangle$; with $a^{-1} ba = b^t$

where t = 97, 226, 648, 772, 792

19. For p = 1093, and for a subgroup of order (7) (1093), we have;

$G_1 \cong \langle a \rangle \times \langle b \rangle$; with $a^{-1} ba = b^t$

where t = 3, 9, 27, 81, 1036

Since $81 = 3^4$, $27 = 3^3$ and $9 = 3^2$, we see that $t_1$, $t_2$, $t_3$ and $t_4$ give rise to the same non-Abelian isomorphic type. Hence we have only two non- Abelian isomorphic types.

20. For p = 1163, and for a subgroup of order (7) (1163), we have;

$G_1 \cong \langle a \rangle \times \langle b \rangle$; with $a^{-1} ba = b^t$

Where t = 44, 383

21. For p = 1933 and for a subgroup of order (7) (1933), we have;

$G_1 \cong \langle a \rangle \times \langle b \rangle$; with $a^{-1} ba = b^t$

Where t = 1000, 1069, 1285

**3.10 LEMMA**

Groups of order n = 7p where p ≡ 1(mod 7) have at most six non-Abelian isomorphic types.

**PROOF:** This is similar to the proof for groups of order 3p and 5p except that t has at most six distinct values $t_1$,$t_2$, $t_3$, $t_4$, $t_5$ and $t_6$. Any other value will be a prime power of one of the $t_i$'s for i = 1, 2, 3, 4, 5, 6.

**3.11 FOR SUBGROUPS OF ORDER 11p**

1. For those primes p such that p ≡ 1 (mod. 11) we give few results of such subgroups of order 11p. We also assume two element generators, a b say, such that $a^{11} = b^p = 1$

   For p = 23 and for subgroups of order (11) (23) we have;

   $G_1 \cong \langle a \rangle \times \langle b \rangle$; with $a^{-1}$ ba = $b^t$

   where t = 2, 3, 12, 13, 18

2. For p = 67 and for subgroups of order (11) (67) we have the following non-Abehian types:

   $G \cong g\langle a \rangle \times \langle b \rangle$; with $a^{-1}$ ba = $b^t$

   where t = 9, 14, 15, 22, 24, 25, 40, 59, 62, 64

3. For p = 331 and for a subgroups of order (11) (331) we have;

   $G \cong \langle a \rangle \times \langle b \rangle$; with $a^{-1}$ ba = $b^t$

   where t = 4, 80, 85, 111, 120, 167, 180, 270, 274, 293

4. For p = 353 and for a subgroups of order (11) (353) we have;

   $G \cong \langle a \rangle \times \langle b \rangle$; with $a^{-1}$ ba = $b^t$

   where t = 22, 58, 131, 140, 185, 187, 217, 231, 256, 337

5. For p = 419 and for a subgroups of order (11) (419) we have;

   $G \cong \langle a \rangle \times \langle b \rangle$; with $a^{-1} ba = b^t$

   where t = 13, 59, 69, 102, 129, 152, 169, 300, 334, 348

6. For p = 463 and for a subgroups of order (11) (463) we have;

   $G \cong \langle a \rangle \times \langle b \rangle$; with $a^{-1} ba = b^t$

   where t = 15, 55, 134, 158, 247, 337, 356, 362, 425

**3.12 LEMMA**

Groups of order 11p where $p \equiv 1 \pmod{11}$ have at most ten non-Abelian isomorphic types.

**PROOF:** This is similar to the proof for groups of order 3p and 5p except that t has at most six distinct values $t_1$,$t_2$, $t_3$, $t_4$, $t_5$ $t_6$, $t_7$, $t_8$, $t_9$, and $t_{10}$ Any other value for t will be a prime power of one of the $t_i$'s for i = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.

**3.13 FOR SUBGROUPS OF ORDER 13p, WHERE p ≡ 1 (mod 13)**

We also assume that such subgroups are generated by two elements a and b such that $a^{13} = b^p = 1$

1. For p = 53 and for subgroups of order (13) (53) we have

   $G \cong \langle a \rangle \times \langle b \rangle$; with $a^{-1} ba = b^t$

   where t = 10, 13, 15, 16, 24, 28, 36, 42, 44, 46, 47, 49

2. For subgroups of order (13) (79) we have

   $G \cong \langle a \rangle \times \langle b \rangle$ with $a^{-1} ba = b^t$

   where t = 8, 10, 18, 21, 22, 38, 46, 52, 62, 64, 65, 67

3. For subgroups of order (13)(131) we have

   $G \cong \langle a \rangle \times \langle b \rangle$; with $a^{-1} ba = b^t$

where t = 39, 45, 52, 60, 62, 63, 80, 84, 99, 107, 112, 113

4. For subgroups of order (13) (443) we have

$G \cong g\langle a\rangle \times \langle b\rangle$; with $a^{-1}ba = b^t$

where t = 35, 38, 56, 135, 184, 188, 238, 339, 347, 356, 378, 383

**3.14 LEMMA**

Groups of order n = 13p for p ≡ 1(mod 13) have at most twelve non-Abelian isomorphic types.

**PROOF:** This is similar to the proof for groups of order 3p and 5p except that t has at most six distinct values $t_1$,$t_2$, $t_3$, $t_4$, $t_5$ $t_6$, $t_7$, $t_8$, $t_9$, $t_{10}$, $t_{11}$, and $t_{12}$. Any other value for t will be a prime power of one of the $t_i$'s for i = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12.

**3.15 GROUPS OF ORDER n = sp WITH NO NON-ABELIAN ISOMORPHIC TYPES.**

We, however, make a comment on why the groups of order n = sp such that p is not congruent to 1 modulo s and reasons why they do not have non-abelian isomorphic types.

To do this a group of order 15 will be considered first.

Let $|G| = 15$ =3x5. G has only one Sylow 5 - subgroup H, say, which is normal in G. Let H and K be cyclic subgroups of order 5 and 3 respectively. We have that

$H \cap K = \{e\}$. Again, any subgroup containing H and K has a multiple of 15. Hence |H x K|=15, i.e. H x K = G.

Therefore, G = H x K implies that $G = C_5 \times C_3 \cong C_{15}$

Hence G is cyclic and therefore Abelian. Supposing a and b are generators of G. Then $ba = ab^t$ where $t \neq 1$ would generate a non-Abelian isomorphic type. This is not possible as none of the values 2, 3 and 4 ensured that $ab^t$ has order 5 or 3.

Note that 5≡2 (mod. 3).

We similarly looked at groups of order 35 = 5 x 7. Again, it is noticed that 7 is congruent to 2 modulo 5 and hence such does not have a non-abelian isomorphic type. A group of order 65 = 5 x 13 has the same behavior as 13 is congruent to 3 modulo 5.

With our scheme we outline the following examples:

For groups of order 5p where $p \equiv k \pmod 5$, $k > 1$ especially where k = 4.

We have the following few values for t:

For subgroups of order 5p we have for $p \equiv 4 \pmod 5$ the following:

1. For subgroup of order (5)(1999) with $a^5 = b^{1999} = 1$, t = 1813;
2. For sub group of order (5)(3079) with $a^5 = b^{3559} = 1$, we have t = 2887;
3. For sub group of order (5)(3559) with $a^5 = b^{3559} = 1$, we have t = 1893

Hence no value of t will ensure closure for $ab = b^t a$

We will also be considering subgroups that are generated by two elements a and b such that $a^7 = b^p = 1$ but p is not congruent to 1 modulo 7.

1. For p = 373 and for a subgroup of order (7) (373), we have t = 259, 281;
2. For p = 401 and for a subgroup of order (7) (401), we have t = 265, 357;
3. For p = 457 and for a subgroup of order (7) (457), we have t = 237, 305, 442;
4. For p = 541 and for a subgroup of order (7) (541), we have t = 463;
5. For p = 653 and for a subgroup of order (7) (653), we have t = 614;
6. For p = 571 and for a subgroup of order (7) (571), we have t = 741;
7. For p = 1283 and for a subgroup of order (7) (1283), we have t = 714, 1097;
8. For p = 1297and for a subgroup of order (7) (1297), we have t = 321;
9. For p = 1493 and for a subgroup of order (7) (1493), we have t = 835, 1205;
10. For p = 1619 and for a subgroup of order (7) (1619), we have

    t = 534, 837, 1359.

11. For p =1787 and for a subgroup of order (7) (1787), we have t = 1100, 1393;

12. For p =1871 and for a subgroup of order (7) (1871), we have t = 478, 667, 806, 1747

13. For p = 1995 and for a subgroup of order (7) (1995), we have t = 1289.

No non-Abelian isomorphic type was obtained due to inability of closure property to be satisfied.

For Primes p such that p ≡ 3 (mod 7) the following values of t were obtained:

1. For p = 521 and for subgroup of order (7) (521) and for $a^7 = b^{521} = 1$, we have t = 345;

2. For p = 647 and for subgroups of order (7) (647) and for $a^7 = b^{647} = 1$, we have t = 259;

3. For p = 829 and for subgroups of order (7) (879) and for $a^7 = b^{829} = 1$, we have t = 337, 826;

4. For p = 997 and for subgroups of order (7) (997) and for $a^7 = b^{997} = 1$, we have t = 730;

5. For p = 1109 and for subgroups of order (7) (1109) and for $a^7 = b^{1109} = 1$, we have t = 946, 989;

6. For p = 1277 and for subgroups of order (7) (1277) and for $a^7 = b^{1277} = 1$, we have t = 838;

7. For p = 1319 and for subgroups of order (7) (1319) and for $a^7 = b^{1319} = 1$, we have t = 727;

8. For p = 1571 and for subgroups of order (7) (1571) and for $a^7 = b^{1571} = 1$, we have t = 397, 985;

9. For p = 1613 and for subgroups of order (7) (1613) and for $a^7 = b^{1613} = 1$, we have t = 1535;

10. For $p = 1669$ and for subgroups of order (7) (1669) and for $a^7 = b^{1669} = 1$, we have $t = 1031, 1100$;

11. For $p = 1697$ and for subgroups of order (7) (1697) and for $a^7 = b^{1697} = 1$, we have $t = 1619$;

12. For $p = 1823$ and for subgroups of order (7) (1823) and for $a^7 = b^{1823} = 1$, we have $t = 695$;

13. For $p = 1879$ and for subgroups of order (7) (1879) and for $a^7 = b^{1879} = 1$, we have $t = 391, 227$;

14. For $p = 1849$ and for subgroups of order (7) (1849) = 1, we have $t = 1340, 1532, 1788$.

Furthermore, for primes, p say, such that $p \equiv 4 \pmod 7$ the following values for are obtained:

1. For $p = 263$ and for subgroups of order (7) (263) and for $a^7 = b^{263} = 1$, we have $t = 225$;

2. For $p = 389$ and for subgroups of order (7) (389) and for $a^7 = b^{389} = 1$, we have $t = 233$;

3. For $p = 487$ and for subgroups of order (7) (487) and for $a^7 = b^{487} = 1$, we have $t = 485$;

4. For $p = 557$ and for subgroups of order (7) (557) and for $a^7 = b^{557} = 1$, we have $t = 433$;

5. For $p = 907$ and for subgroups of order (7) (907) and for $a^7 = b^{907} = 1$, we have $t = 687, 786$;

6. For $p = 1481$ and for subgroups of order (7) (1481) and for $a^7 = b^{1481} = 1$, we have $t = 1361$;

7. For $p = 1831$ and for subgroups of order (7) (1831) and for $a^7 = b^{1831} = 1$, we

have t = 1578;

8. For p = 1901 and for subgroups of order (7) (1901) and for $a^7 = b^{1901} = 1$, we have t = 618;

9. For p = 1999 and for subgroups of order (7) (1999) and for $a^7 = b^{1999} = 1$, we have t = 1033, 1156, 1409.

For Primes p such that p ≡ 5 (mod 7), the following values of t which equally failed the closure property were obtained:

1. For p = 313 and for subgroups of order (7) (313) and for $a^7 = b^{313} = 1$, we have t = 197;

2. For p = 439 and for subgroups of order (7) (439) and for $a^7 = b^{439} = 1$, we have t = 315;

3. For p = 523 and for subgroups of order (7) (523) and for $a^7 = b^{523} = 1$, we have t = 402, 479;

4. For p = 593 and for subgroups of order (7) (593) and for $a^7 = b^{593} = 1$, we have t = 521;

5. For p = 677 and for subgroups of order (7) (677) and for $a^7 = b^{677} = 1$, we have t = 395, 610;

6. For p = 1789 and for subgroups of order (7) (1489) and for $a^7 = b^{1489} = 1$, we have t = 341;

7. For p = 1559 and for subgroups of order (7) (1559) and for $a^7 = b^{1559} = 1$, we have t = 715;

8. For p = 1951 and for subgroups of order (7) (1951) and for $a^7 = b^{1783} = 1$, we have t = 433;

For those Primes p in the Congruence Class of 6 modulo 7 the following values for t were obtained for $a^7 = b^p = 1$:

1. For p = 223 and for subgroups of order (7) (223) we have t = 197;
2. For p = 461 and for subgroups of order (7) (461) we have t = 355;
3. For p = 587 and for subgroups of order (7) (587) we have t = 443;
4. For p = 601 and for subgroups of order (7) (601) we have t = 513;
5. For p = 769 and for subgroups of order (7) (769) we have t = 683;
6. For p = 1693 and for subgroups of order (7) (1693) we have t = 683, 1292;
7. For p = 1777 and for subgroups of order (7) (1777) we have t = 213;
8. For p = 1847 and for subgroups of order (7) (1847) we have t = 608, 926;
9. For p = 1889 and for subgroups of order (7) (1889) we have t = 386;
10. For p = 1973 and for subgroups of order (7) (1973) we have t = 1972;

From the examples outlined above, we state the following:

**3.16 LEMMA**

Any group of order n = sp where p is not congruent to 1 modulo s does not have a non-Abelian isomorphic type since none of the values for t can satisfy closure property for $ba = ab^t$ as $t^s$ is not congruent to 1 modulo p. Hence there cannot be a non-Abelian isomorphic type.

In considering groups of order n = s(pq), groups of order 30, 42, and 70 were first treated. For consistency, x, y and z were used as generators and $z^{-1}xyz = (xy)^t$ where different values of $t \neq 1$ will give different non-Abelian isomorphic types that can be obtained.

For The Non-Abelian Types of Groups of Order 30, it can be seen from that

30 = 5 x 3 x 2. Since every finite Abelian group is a direct sum of primary cyclic groups there exists only one type of Abelian group of order 30 and this type is

necessarily cyclic by Theorem (1.4.11). For the case where G is non-Abelian and of order 30, by Theorem (1.4.10), G has 1 or 6 subgroups of order 5 and 1 or 10 subgroups or order 3. It is obvious that a group of order 30 cannot have 6 subgroups of order 5 and 10 subgroups of order 3 at the same time.

Hence any group of order 30 must have either its Sylow 5 - subgroup or its Sylow 3 - subgroup normal in G.

Hence if $H = \langle x \rangle : x^5 = 1$ and

$H = \langle y \rangle : y^3 = 1.$

Either H or K is normal in G. Hence

$$HK = KH$$

is a subgroup of G.

By factor theorem, $|HK| = 15.$ Since any group of order 15 is Abelian and by (2.2), it follows than that $xy = yx$.

Hence, we look at the situation where

$G = \langle xy, z \rangle : (xy)^{15} = 1, z^2 = 1$

Since the subgroup $HK = \langle xy \rangle$ has index 2 in G, it must therefore be normal in G.

Hence $z\,x\,yz = (xy)^t$,

where $t^2 \equiv 1 \pmod{15}$

and t = 1, 4, 11,and 14.

Since t = 1 implies that G is Abelian, we start from t = 4.

Therefore the following isomorphic type results

(i) $G_1 = \langle xy,z \rangle, (xy)^{15} = 1, z^2 = 1, zxyz = (xy)^4 = x^{-1}y$

$$= \langle x,y,z \rangle, x^5 = y^3 = z^2 = 1, zxz = x^{-1}$$

$xy = yx, yz = zy$

$= \langle x,z \rangle \times \langle y \rangle$

$= D_5 x C_3$

Setting t = 11 the following relations are obtained:

(ii) $G_2 = \langle xy,z \rangle, (xy)^{15} = 1, z^2 = 1, zxyz = (xy)^{11} = xy^{-1}$

$= \langle x,y,z \rangle, x^5 = 1, y^3 = 1, z^2 = 1, xy = yx, zyz = y^{-1}$

$= \langle x,yz \rangle, x^5 = 1, y3 = 1, z^2 = 1, xy = yx, zyz = y^{-1}$

$= \langle x \rangle \times \langle y,z \rangle = gp\{x\} \times gp\{y,z\}$

$= C_5 x D_6 = C_5 x S_3$.

Finally for t = 14, we again have the relations:

(iii) $G_3 = \langle xy, z \rangle, (xy)^{15} = 1, z^2 = 1, zxyz = (xy)^{-1}$

$= \langle x,y,z \rangle, x^5 = 1, y^3 = 1, z^2 = 1, zxz = x^{-1}, xy = yx, zyz = y^{-1}$

It is observed from the first representation that $G_3$ is the dihedral group $D_{15.}$

The defining relations show that the Sylow 3 - subgroups and Sylow 5 - subgroups are always normal in any group of order 30.

For groups of order 42 and from the factorization

42 = 7 x 3 x 2, it can be seen that the Sylow 7 - subgroup is normal in G by Theorem (1.4.9).

Here, $H = \langle x \rangle : x^7 = 1$

$K = \langle y \rangle : y^3 = 1$

and $HK = KH \; and |HK| = 21$.

But $HK = \langle xy \rangle$ has index 2 in G, it must be normal in G. For more than one subgroups H we have for

$z\,xyz = (\,xy)^{t}$

where $t^{2} \equiv 1 \pmod{21}$

Hence t = 1, 8, 13 and 20.

t = 1is trivial and we look at the rest.

For t = 8, we have the following isomorphic type

(i) $G_1 = \langle xy, z\rangle, (xy)^{21} = 1, z^{2} = 1, zxyz = xy^{\;1}$

$= \langle x, y, z\rangle, x^{7} = 1, y^{3} = 1, z^{2} = 1, xz = zx, xy = yx, zyzy = y^{\;1}$

$= \langle x, z\rangle \times \langle y\rangle$

$= D_6 \times C_7$

For t = 13 we obtain the following:

(ii) $G_2 = \langle x, y, z\rangle, x^{7} = y^{3} = z^{2} = 1, zxyz = (xy)^{13} = x^{6}y = x^{\;1}y$

$= \langle x,y,z\rangle, x^{7} = y^{3} = 1 = z^{2}, zyz = y, xy = yx, zx = x^{-1}$

$= D_7 \times C_3$.

Finally, for t = 20 we obtain the following:

(iii) $G_3 = \langle xy, z\rangle, x^{7} = 1, y^{3} = 1, z^{2} = 1, zxyz = (xy)^{20} = x^{6}y^{2} = x^{-1}y^{-1}$

$= \langle x,y,z\rangle, x^{7} = 1, y^{3} = 1, z^{2} = 1, zxz = x^{-1}, zyz = y^{-1}$

From the representation of above, $G_3$ is the dihedral $D_{21}$

Next groups of order 70 were considered as follows.

Since 70 = 7 x 5 x 2, it should be seen that there exists only one class of Abelian group or order 70 which is necessarily cyclic.

By a similar approach, we see that any group of order 70 must have either its Sylow 7 - subgroup or its Sylow 5-subgroup normal in G.

Hence,

$H = \langle x \rangle : x^7 = 1$ and

$K = \langle y \rangle : y^5 = 1$

For, $zxyz = (xy)^t$ and $t^2 \equiv 1 \pmod{35}$ we obtain t = 1, 6, 29, 34.

For t = 6 we have:

(i) $G_1 = \langle x, y, z \rangle, (xy)^{35} = 1 \ , z^2 = 1, xy = yx, zy = yz, zxz = x^{-1}$

$= \langle x,z \rangle \times \langle y \rangle, \ x^7 = 1, y^5 = 1, z^2 = zxz = x^{-1}, zy = yz$

$= D_7 x C_5$

If t = 29, we have

(ii) $G_2 = \langle xy, z \rangle, (xy)^{21} = 1 \ , z^2 = 1, xy = yx, zx = xz, zyz = y^{\ 1}$

$= \langle x \rangle \times \langle y, z \rangle$

$= C_7 x D_5 = C_7 x S_3.$

If t = 34, we have:

(iii) $G_3 = \langle x, y, z \rangle, (xy)^{35} = 1, z^2 = 1, zxz = x^{\ 1}, zyz = y^{\ 1}$

$G_3$ is here dihedral group of order 70, i.e. $D_{35}$.

## 3.18 SUMMARY OF DEFINING RELATIONS

For groups of order 30, we have

(i) $G_1 = \langle a \rangle, a^{30} = 1$

(ii) $G_2 = \langle a,b \rangle, a^{15} = 1, b^2 = 1, bab = a^4$

$= \langle x,y,z \rangle, x^5 = 1, y3 = 1, z^2 = 1, xy = yx, zx = xz, zyz = y^{-1}$

$= \langle x, z \rangle \times \langle y \rangle$

(iii) $G_3 = \langle a,b \rangle, a^{15} = 1, b^2 = 1, bab = a^{11}$

$= \langle x,y,z \rangle, x^5 = 1, y^3 = 1, z^2 = 1, xy = yx, zx = xz, zyz = y^{-1}$

$= \langle x \rangle \times \langle y,z \rangle.$

(iv) $G_1 = \langle a,b \rangle$, $a^{15} = 1$, $b^2 = 1$, $bab = a^{-1}$

For groups of order 42, we have

(i) $G_1 = \langle a \rangle$, $a^{42} = 1$.

(ii) $G_2 = \langle x \rangle \times \langle y,z \rangle$,

$$x^7 = 1, y^3 = 1, z^2 = 1, xz = zx, \ xy = yx, zyz = y^{-1}$$

(iii) $G_3 = \langle x,y,z \rangle$, $x^7 = 1$, $y^3 = 1$, $z^2 = 1$, $zxz = x^{-1}$, $zy = yz$ .

(iv) $G_4 = \langle x,y,z \rangle$, $x^7 = 1$, $y^3 = 1$, $z^2 = 1$, $zxz = x^{-1}$ , $zyz= y^{-1}$.

For groups of order 70, we have

(i) $G_1 = \langle xy,z \rangle$, $(xy)^{35} = 1$, $z^2 = 1$.

(ii) $G_2 = \langle x, z \rangle \times \langle y \rangle, x^7 = 1, y^5 = 1, z^2 = 1, zxz = x^{\ 1}, zy = yz, xy = yx$

(iii) $G_3 = \langle x \rangle \times \langle y, z \rangle, x^7 = 1, y^5 = 1, z^2 = 1, zxz = x \ , zyz = y^{\ 1}, xy = yx$

(iv) $G_4 = \langle xy,z \rangle$, $(xy)^{15} = 1$, $z^2 = 1$, $zxz = x^{-1}$, $zyz = y^{-1}$ .

The above results can be summarized as a proposition:

**3.19 PROPOSITION**

There are three non-Abelian isomorphic types of groups of order $n = spq$, $s<p<q$. ($n=30$, 40 and 70)

$G_1 = \langle a \rangle$; $a^{spq} = 1$

the cyclic group which is Abelian

$$G_2 = \langle x, y, z \rangle; x^q = y^p = z^s = 1; xy = yx, z^{-1}yz = y, z^{-1}xz = x^{t_1}$$

where $t_1=p+1$. This is the case for groups of order 30 and 70.

$G_3 = \langle a,b \rangle$; $a^{pq} = b^s = 1$, $b^{-1}ab = a^{t_2}$ where $t_2 = pq - 1$

This is generally obtained for groups of order 30, 42 and 70 respectively.

$G_4$ = ‹ x,y,z ›; xq = y p = zs = 1; xz = zx, xy = yx,

$$z^{-1}yz = y^{t_3}, \text{ where} = t_3 = (p-1)q+1.$$

This was again seen to be true for groups of order 30 and 70.

With our scheme, we list the possible values of t which gave rise to non-Abelian isomorphic types of groups of order n =2pq:

For n = 154 = 2 x 7 x 11 = 2 x 77; t = 34, 43 and 76.

For n = 182 = 2 x 7 x13 = 2 x 91; t = 27, 64 and 90.

For n = 238 = 2 x 7 x 17 = 2 x 119; t = 50, 69, and 118.

For n = 442 = 2 x 13 x 17 = 2 x 221; t = 103, 118, and 220.

For n = 494 = 2 x 13 x 19 = 2x247; t = 77, 170, and 246.

For n = 266 = 2x 7 x 19 = 2 x 133; t = 20, 113, 132.

For n = 286 = 2 x 11 x 13 = 2 x 143; t = 12, 131, and 142.

For n = 374 = 2 x 11 x 17 = 2 x 187; t = 67, 120, and 1186.

For n = 418 = 2 x 11 x 19 = 2 x 209; t = 56, 153, and 208.

For n = 66 = 2 x 3 x11 = 2 x 33; t = 10, 23, and 50.

For n = 102 = 2 x 3 x 17 = 2 x 51; t = 16, 35, and 50.

For n = 114 = 2 x 3 x 19 = 2 x 57; t = 20, 37, and 56.

For n = 110 = 2 x 5 x 11 = 2 x 55; t = 21, 34, and 54.

For r =130 = 2 x 5 x 13 = 2 x 65; t = 14, 51, and 64.

For n = 170 = 2 x 5 x 17 = 2 x 85; t = 16, 69, and 84.

For n = 190 = 2 x 5 x 19 = 2 x 95; t = 39, 56, and 94.

For n = 230 = 2 x 5 x 23 = 2 x 115; t = 24, 91, and 114.

**3.20 Lemma**

Groups of order n = 2pq has at most three non-Abelian isomorphic types.

≥ 3 were considered.

**PROOF:** This is similar to the proof of Lemma 3.21 except that $t = t_i$, $1 \leq i \leq 8$.

# CHAPTER FOUR
# RESULTS

Here we put up our examples and findings from the previous chapter.

**4.1 RESULT 1.** Groups of order 2p have only one non-Abelian Isomorphic type.

**PROOF:**

Let G = ‹ a › × ‹ b › such that $a^2 = b^p = 1$. Then the non-Abelian isomorphic type must have the relation

$$ab = b^t a,$$

where $1 < t < p$.

We need to show that only one value of t satisfies the above defining relationship.

First, we notice that if t = 2 then

$$2^2 \equiv 1 \pmod 3) \text{ and we see that } 2^2 - 1 = 3.$$

This is true for p = 5, 7, 11, 13,...,

That is if t = 4, then

$$4^2 = 16 \equiv 1 \pmod 5.$$

Also for t = 6, 10, 12,…., for p = 7, 11, 13, …..

Hence for any prime p > 2, we show that

$$(p - 1)^2 \equiv 1 \pmod p$$

$$\Rightarrow (p - 1)^2 - 1 = kp \text{ for some integer } k.$$

$$\Rightarrow p^2 - 2p + 1 - 1 = p^2 - 2p = p(p - 2) = kp$$

Where k = p – 2 which is an integer.

Hence for any group of order 2p, there is only one non-Abelian isomorphic type with the defining relation

$$ab = b^t a$$

and t will take value p – 1 as the only possibility.

## 4.2 RESULT II

Groups of order 3p have at most two non-abelian isomorphic types.

**PROOF:**

Let $G = \langle a \rangle \times \langle b \rangle$ such that $a^3 = b^p = 1$. The non-Abelian isomorphic types must have the relations.

(i) $ab = b^{t_1}$

(ii) $ab = b^{t_2}$

where $t_1$ and $t_2$ are not powers of each other.

Our problem here is to determine that there are two distinct values of t in the interval

$$1 < t < p,$$

which satisfy the defining relationship $ab = b^t a$.

Here, we have

$$t^3 \equiv 1 \pmod{p}$$

$\Rightarrow$ $t^3 - 1 = kp$ for some integer k, and $t^3 - kp - 1 = 0$ is a polynomial of degree 3 and would have at most three distinct roots.

By the examples of the non-Abelian isomorphic types of groups of order n = 3p, t will take values from 2, 3,…, p – 1.

From our examples above and Lemma 3.3 and 3.4, we see that only two values of t satisfied our requirement. We denote these values by $t_1$ and $t_2$.

## 4.3 RESULT III

Groups of order 5p have at most four non-abelian isomorphic types.

**PROOF:**

For $G = \langle a \rangle \times \langle b \rangle$, with $a^5 = b^p = e$, the non-Abelian isomorphic types are of the form

$ba = ab^{t_i}$

where i = 1, 2, 3, 4.

As we have shown in the proof of non-Abelian isomorphic types of groups of order 3p and from examples 3.5 and 3.7, $t_i$'s are within the interval $1 < t < p$ hence the theorem.

## 4.4 MAIN RESULT

There are more than one non-Abelian isomorphic types of groups of order $n = sp$, where $(s,p) = 1$.

## 4.5 MOTIVATION

Dihedral group is a family of symmetry groups which are not commutative. When we consider a triangular plate we can have six rotational symmetries (with r and s as rotations) which are

$e, r, r^2, s, rs, r^2s$

The above six elements form a group denoted by $D_3$ . As an illustration

$sr^2 = s(rr) = (sr) r = (r^2s) r = r^2(sr)$

$= r^2(r^2s) = r^4s = r^3(rs) = e(rs)= rs.$

Notice that associative law was repeatedly used. The dihedral group $D_n$ is the rotational symmetry group of the plate with n equal sides. Its elements can be described in the same manner as that used for $D_3$. If r is a rotation of the plate through $\frac{2\pi}{n}$ about the axis of symmetry perpendicular to the plate, and s a rotation through $\pi$ about an axis of symmetry which lies in the plane of the plate, we have the following elements of $D_n$

$e, r, r^2, \ldots, r^{n-1}, s, rs, r^2s, \ldots , r^{n-1}s.$

Clearly,

$r^n = e, s^2 = e$

and geometrically

$sr = r^{n-1}s$ and since $r^{n-1} = r^{-1}$, it is usual to write $sr = r^{-1}s$ (Armstrong M.A.).

This is obtained in the situation where the order of a group n is 2p where p is a prime. This also matches the situation where t is determined for two element generators, a and b with $a^{-1}ba = b^t$ and $t^2 \equiv 1 \pmod p$, where p is the order of b and $a^2 = 1$.

Here we cite simple examples of groups of order 6 = 3x2, 14 = 7x2, and so on. In the situation where a group of order 15 = 3x5 is considered, there was value of t satisfying

$a^{-1}ba = b^t$ with $t^3 \equiv 1 \pmod 5$.

For the group of order 21 = 3x7 it is easy to see that

$a^{-1}ba = b^2$

Here we see t taking a value which is different from p-1.

This process continues but as the primes s and p become bigger, with p > s and s >2, we start noticing for $a^s = b^p = 1$,

and

$a^{-1}ba = b^t$, that t can assume several values.

Our previous examples showed that we can have more than one value of t satisfying

$t^s \equiv 1 \pmod p$

and non of such values is a power of the other. This informs that a group of order n = sp may have more than one non-Abelian type depending on the number of different values of t that can be determined.

**PROOF (OF THE MAIN RESULT)**

Let G be a group of order n = sp, with (s,p) = 1 and s<p. By Sylow's Theorem there must be only one Sylow p-subgroup in G. This subgroup

$K = \langle b \rangle, b^p = 1,$

which must be normal in G.

Moreover, any other Sylow subgroup must be of the form

$H = \langle a \rangle, a^s = 1.$

Since $K \lhd G$ and we have

$a^{-1}ba \in K$ and

$a^{-1}ba = b^t$

for some integer t.

Clearly, if t = 1, we have that G is Abelian and so ab = ba.

If $p \equiv 1 \pmod s$ then there are s Sylow p-subgroup and we have for $t \neq 1$, that

$$a^{-1}b^k a = (a^{-1}ba)^k = b^{t^k}$$

That is

$$a^{-2}ba^2 = a^{-1}(a^{-1}ba)\, a = a^{-1}b^t a = b^{t^k}$$

this will be repeated up to

$a^{-j}ba^j = b^{t^j}$ for some integer j.

If j = s then the above relation relation yields

$$b = a^{-s}b^t a^s = b^{t^s},$$

we deduce that $p|(t^s - 1) \Rightarrow t^s \equiv 1 \pmod p$

Hence $t^s - 1 = kp$ for some integer k.

Therefore

$$t^s = kp+1$$

and $t = (kp + 1)^{1/s}$

From our examples, if s = 2, we have one value for t.

If s = 3, we have at most two values for t.

Since t takes values in the interval $1<t<p$ which also satisfies the congruence

$t^s \equiv 1 \pmod p$. We denote these values by $t_1, t_2, t_3, \ldots$, where $t_1 \neq t_2 \neq t_3 \neq \ldots$

and none is a prime power of the other. We have the following possibilities

$a^{-1}ba = b^{t_1}, a^{-1}ba = b^{t_2}, a^{-1}ba = b^{t_3}, \ldots$

It is obvious that $b^{t_1} \neq b^{t_2} \neq b^{t_3} \neq \ldots$

Hence by Theorems 3.6, 3.10, and examples 3.4, 3.7, 3.8, 3.12, 3.13 and 3.14 we have determined different values of t which gave rise to different non-Abelian isomorphic types.

**4.6 COROLLARY**

Only one value of t satisfies the congruence $t^2 \equiv 1 \pmod p$ where $(2,p) = 1$ and p is a prime.

**PROOF**

Obviously p divides $t^2 - 1$ which implies that $t^2 = kp + 1$, for some integer k. By choosing the possible values of t in the interval $1 < t < p$, we need to show that only one value of r satisfies the congruence $t^2 \equiv 1 \pmod p$.

From (4.1) this value is p -1. That is,

$(p - 1)^2 = p^2 - 2p + 1$. Hence p divides $p^2 - 2p$.

If on the contrary $t = p - k$, where $k > 1$ and $k < p - 1$, then

$(p-k)^2 = p^2 - 2kp + k^2$ .

This is not a multiple of p.

## 4.7 RESULT FOR GROUPS OF ORDER n = sp SUCH THAT P IS NOT CONGRUENT TO 1 MODULO s.

The groups of order n = sp such that p is not congruent to 1 modulo s cannot have non-abelian isomorphic type.

**PROOF:**

For G = ‹ x,y ›, $x^s = y^p = 1$, since $yx \neq xy$, then $yx = y^t x$, for some inter t > 1.

So,

$$x^{-1} y x = y^t,$$

for some t in the interval 1 < t < p, will have order s or p. No such r satisfies the closure property of such groups. Hence groups of order n = sp such p is not congruent to 1 modulo s does not have a non-abelian isomorphic type. Hence such groups are necessarily cyclic.

This affirms the assertion that:

"There is just one group of order n if and only if n is a product of distinct primes $p_1, p_2, p_3, ..., p_k$ such that $p_j$ does not divide $(p_i - 1)$ for $1 \leq i \leq k$, $1 \leq j \leq k$"

(John R. Durbin).

The above conclusion was reached after considering the isomorphic types of groups of order n for each n from 1 to 32.

Later on, we will see the extent of the truth of the above assertion when groups of order n factorizable into a product of three primes are considered.

For groups of order n = spq where s, p and q are distinct primes we have the following result:

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$H=\langle x\rangle, x^{q}=1$, or $K=\langle y\rangle, y^{p}=1$

$H \lhd G$ or $K \lhd G$

$|KH| = pq.$

$\therefore G=\langle xy, z\rangle, (xy)^{pq}=z^{s}=1$

$z^{-1}xyz=(xy)^{t}$

$pq \,|\, (t^{s}$

$1<t\le pq$

$t^{s}$

$t^{s}=kpq+1$

$G_{r_i} \quad \langle xy, z\rangle, (xy)^{pq}=1, z^{s}=1;$

$z(xy)z^{-1}=(xy)^{t_i}$

⑥④ ☝♐④④♋

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# CHAPTER FIVE
# CONCLUSION AND RECOMMENDATIONS

Our work here was organized in the following manner: First we looked at groups of order 2p, where p is a prime. Since every positive prime is congruent to 1 modulo 2, we did not have much difficulty in out lining the nature of the non-abelian groups of such orders. Next, we used our scheme to look at groups of order n = sp in which case we particularly looked at those prime greater 3 and are congruent to 1 modulo 3. We also tried to display their defining relation in most of the cases.

Armed with our scheme, we also sort for and obtained the number of non-abelian isomorphic types of groups of order 5p, 7p, 11p, 13p and so on. We kept the demand that p is congruent to 1 modulo 5, 7, 11, 13, in all the cases.

From the group of order 15 = 3 x 5, we sought to see what would be the fate of groups whose prime factorization were such that none of the factors if congruent to one modulo the other.

For groups of order n = spq, where s, p, and q are distinct primes, we first considered groups order 30, 42, and 70. One readily observes that such groups are of the form 2pq where each of p and q is congruent to 1 modulo 2 but may not be congruent to 1 modulo each order. We later considered when $s \neq 2$. The demand here is not restricted to each of the primes being congruent to 1 modulo others.

## 6.1 SUMMARY OF RESULTS

The area of group classification up to isomorphism and determination of isomorphic types of groups of certain orders is as old as group theory itself.

There is no easy way out hence many tend to pursue it through different approaches. In this Thesis we devoted our work to finding the non-abelian isomorphic types of certain groups of order n = sp, spq and found the following:

1. We developed a scheme that determines the numbers that help to forms the non-Abelian isomorphic types of a group can be.
2. We gave with examples proofs of the form of the non-abelian isomorphic types of groups of order 2p, 3p, 5p, 7p,…., and 2pq, 5pq, 7pq,…

## 5.2 CONTRIBUTION TO KNOWLEDGE

(i) That the number of the non-abelian isomorphic types of groups of order n = sp increase as the values of s and p increase.

(ii) Why groups of order n = sp, where p is not congruent to 1 modulo s, cannot have a non-abelian isomorphic type.

(iii) That groups of order n = spq have non-abelian isomorphic type irrespective of whether the prime factors are congruent to 1 modulo others, that is whether s divides p -1 and q -1.

(iv) That the relationship between the prime factors of the order of groups determine to a large extent whether such groups would have non-abelian isomorphic type or not.

## 5.3 AREAS OF FURTHER RESEARCH

1. There is room to further look at groups whose orders are factorizable into more that three factors.
2. The use of those groups whose prime factors s and p such that p is not congruent to 1 modulo s.
3. The possibility of the use of isomorphic types to resolve the fundamental relationship between the underlying biochemistry and the structure of erythrocyte and other cells.
4. To determine the relationship existing between the different values of r and the

prime p in the non-Abelian isomorphic types of groups of order 3p, 5p, 7p and so on.

5. To determine the non-Abelian isomorphic types of groups of order n = 11pq, 13pq and so on where p, q > 13.

## 5.4 CONCLUSION

Based on our finding so far we showed that the number of non-abelian groups of order n = sp increase as s and p increase for p congruent to 1 modulo s in all the cases. Again, we see that for n = spq, the non-abelian isomorphic types do increase as s, p and q becomes larger due possibly to the congruent relationship among the prime factors.

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