SCIENCE FORUM (JOURNAL OF PURE AND APPLIED SCIENCES) 21 (2021) 416 – 424

http://dx.doi.org/10.5455/sf.115179

Science Forum (Journal of Pure and Applied Sciences)

journal homepage: www.atbuscienceforum.com

Reformulated Adomian decomposition method for the approximation of special linear and nonlinear two-point boundary value problems

Joshua Sunday[1,*], Joshua A. Kwanamu[2], Nathaniel M. Kamoh[1], Yusuf Skwame[2]

[1]*Department of Mathematics, University of Jos, Jos, Nigeria*

[2]*Department of Mathematics, Adamawa State University, Mubi, Nigeria*

ABSTRACT

Boundary value problems (BVPs) of higher order have been found to be potentially applicable in hydro-magnetic stability, hydrodynamics, chemical reactions, heat power transmission theory, and the boundary layer theory in fluid mechanics. In this research, a method which decomposes the solution into the series which converges rapidly shall be derived. We shall call this method the reformulated Adomian decomposition method (RADM). This method is an improvement over Aadomian decomposition method (ADM). The RADM is derived in such a way that on imposing the boundary conditions on the approximant, a system of equations is obtained which in turn is solved for the undetermined constants. On substituting the resulting constants into the solution function, we obtain a series solution to the problem. The RADM is applied on some linear and nonlinear two-point BVPs and from the results obtained, the method is said to be computationally reliable.

ARTICLE INFO

Article history:
Received 14 June 2020
Received in revised form
12 May 2021
Accepted 13 May 2021
Published 12 August 2021
Available online 12 August 2021

KEYWORDS

Approximation
Two-point BVPs
Linear
Nonlinear
RADM

1. Introduction

The Aadomian decomposition method (ADM) is a powerful technique which provides efficient algorithms for analytic approximate solutions and numeric simulations of linear and nonlinear equations, including ordinary differential equations, partial differential equations, integral equations, integro-differential equations (Duan et al., 2013).

The ADM permits us to solve both linear and nonlinear boundary value problems (BVPs) without unphysical restrictive assumptions such as required by linearization, perturbation, *ad hoc* assumptions, guessing the initial term or a set of basis functions and so on. Furthermore, the ADM does not require the use of Green's functions which are not easily determined in most cases.

However, one of the shortcomings of the ADM is that the Adomian polynomial depends heavily on $y_0(x)$ which sometimes affect the convergence of the solution series and makes the computation tedious (especially when the problem is nonlinear in nature). It is in view of this that we are motivated to modify the ADM to obtain a method called the reformulated Aadomian decomposition method (RADM).

In this paper, our aim is to find approximate solution to special linear and nonlinear two-point BVPs of the form,

$$y^{(2m)}(x) = f(x, y), 0 < x < a \tag{1}$$

subject to the boundary conditions,

$$y^{(2i)}(0) = \rho_{2i},\ y^{(2i)}(1) = \xi_{2i},\ i = 0,1,2,\dots,(m-1) \tag{2}$$

using the RADM. We assume that $y(x)$ and $f(x, y)$ are real and differentiable as many times a required for x€ $[0, a]$. Also, ρ_{2i} *and* ξ_{2i}, $i = 0,1,2,\dots,(m-1)$ are real finite constants (Wazwaz, 2000).

*** Corresponding author** J. Sunday ✉ sundayjo@unijos.edu.ng, joshuasunday2000@yahoo.com ✉ Department of Mathematics, University of Jos, Jos, Nigeria.

A lot of methods have been proposed by different authors for the solution of BVPs. These methods include: extended ADM (Jang,2008); Exp-function method (Mohyud-Din et al., 2009); power series method (Ibijola and Sunday, 2010); optimal homotopy asymptotic method (Javed et al., 2010); B-spline collocation method (Rashidinia and Ghasemi, 2011); shooting method (Sivakumar and Baiju, 2011); two-step method (Phang et al., (2017); diagonal block method (Nadirah et al., 2018); Runge–Kutta method (Akram and Aslam, 2018); Modified ADM (Duan et al., 2013, Opanuga et al., 2017; Waleed, 2011), among others.

2. Derivation of the RADM

In deriving the RADM, we assume that the special BVP in Equation. (1) can be written in form of a general equation as

$$Ly + Ry + Ny = g \tag{3}$$

where y is the system output (unknown function), L is the linear differential operator to be inverted which is just the highest order differential operator, R is the linear remainder operator, N is the nonlinear operator which is assumed to be analytic, and g is the system input (function source).

Let the operator L be defined as

$$L = \frac{d^{2m}}{dx^{2m}} \tag{4}$$

Then the inverse operator L^{-1} is taken to be a $2m-$fold integral operator. Operating L^{-1} on both sides of Equation. (3), we obtain

$$y = L^{-1}(g - Ry - Ny) = L^{-1} f(x, y) \tag{5}$$

Let ρ_{2j+1}, $j = 0,1,2,\ldots,(m-1)$ be the constants of the boundary conditions at derivatives of odd-order defined by

$$\left.\begin{aligned} \rho_1 &= y'(0) \\ \rho_3 &= y'''(0) \\ \rho_5 &= y^{v}(0) \\ &. \\ &. \\ &. \\ \rho_{(2m-1)} &= y^{(2m-1)}(0) \end{aligned}\right\} \tag{6}$$

These constants will be explicitly determined later by applying the boundary conditions at $x = a$ (Wazwaz, 2000). Note that the remaining constants $\rho_0, \rho_2, \rho_4, \ldots, \rho_{(2m-2)}$ are described by boundary conditions at derivatives of even-order, see Equation. (2).

Equation (5) can, therefore, be written as

$$y(x) = \sum_{k=0}^{2m-1} \rho_k \frac{x^k}{k!} + L^{-1}\left(g - Ry - Ny\right) \tag{7}$$

The ADM assumes the solution $y(x)$ of the BVP Equation (1) by an infinite series of the form,

$$y(x) = \sum_{n=0}^{\infty} y_n(x) \tag{8}$$

where the components $y_n(x)$ are determined recursively. The ADM also assumes that the nonlinear function $g - Ry - Ny$ can be decomposed by an infinite polynomial series as

$$g - Ry - Ny = \sum_{n=0}^{\infty} A_n \tag{9}$$

where A_n are the Adomian's polynomials defined by $A_n = A_n(y_0, y_1, y_2, \ldots, y_n)$. Adomian (1994) set the algorithms for computing A_n's for various classes of nonlinearity. Wazwaz (2000) also developed a new algorithm for computing these polynomials.

Putting Equations. (8 and 9) in Equation (7) gives

$$\sum_{n=0}^{\infty} y_n(x) = \sum_{k=0}^{2m-1} \rho_k \frac{x^k}{k!} + L^{-1}\left(\sum_{n=0}^{\infty} A_n\right) \tag{10}$$

Unlike the ADM that uses the recursive relation,

$$\left.\begin{aligned} y_0(x) &= \sum_{k=0}^{2m-1} \rho_k \frac{x^k}{k!} \\ y_{n+1}(x) &= L^{-1}(A_n),\ n \ge 0 \end{aligned}\right\} \tag{11}$$

for the determination of the components $y_n(x)$ of $y(x)$, the RADM will use the relation,

$$\left.\begin{aligned} y_0(x) &= \rho_0 \\ y_1(x) &= \sum_{k=1}^{2m-1} \rho_k \frac{x^k}{k!} + L^{-1}(A_0) \\ y_{n+1}(x) &= L^{-1}(A_n),\ n \ge 1 \end{aligned}\right\} \tag{12}$$

The reason for this reformulation is that Adomian polynomials A_n depend heavily on $y_0(x)$. It is important to state here that the choice of $y_0(x)$ to contain minimal number of terms facilitates the computational behavior of A_n.

Next, we determine approximations to ρ_{2j+1}, $j = 0,1,2,\ldots,(2m-1)$. We then impose the boundary conditions at $x = a$ on the approximant

$$\mu_n = \sum_{m=0}^{n-1} y_m \tag{13}$$

to obtain a system of equations which in turn is solved to obtain the constants ρ_{2j+1}, $j = 0,1,2,\ldots,(m-1)$. On the determination of these constants, the solution to the special linear and nonlinear BVPs of the form (1) is obtained by substituting the components into Equation. (8).

3. Analysis of the Existence and Uniqueness of Solutions of BVP

According to Butcher (2008), three attributes of a BVP that have to be taken into account are whether there actually exists a solution, whether the solution, if it exists, is unique, and how sensitive the solution is to small perturbation of the boundary information.

Thus, in this research, we shall adopt a new way of establishing these results. For a continuous function $f(x, y)$ defined in the region (rectangle)

$$R = \{(x, y): x_0 - \delta < x < x_0 + \delta,\ y_0 - \varepsilon < y < y_0 + \varepsilon\} \tag{14}$$

containing (x_0, y_0), we obtain the following results:

(i) Existence of Solution: there exists a solution $y(x)$ to the BVP Equation (1) defined for $x_0 - \delta_1 < x < x_0 + \delta_1$ (with δ_1 possibly smaller than δ)

(ii) Uniqueness of Solution: for the continuous functions $f(x, y)$ and $\frac{\partial f}{\partial y}(x, y)$ defined in the region (rectangle) R above; there is a unique solution $y(x)$ to the BVP Equation (1) for $x_0 - \delta_2 < x < x_0 + \delta_2$ (with δ_2 possibly smaller than δ_1)

(iii) Boundedness: it is important to state that $f(x, y)$ and $\frac{\partial f}{\partial y}(x, y)$ must be bounded in R. That is, there exist non-negative constants K *and* L such that,

$$|f(x, y)| \le K \tag{15}$$

$$\left|\frac{\partial f}{\partial y}(x, y)\right| \le L \tag{16}$$

Theorem 3.1 (Henrici, 1962)

Suppose that *f* satisfies the assumptions in Equationsss. (5 and 6). Let,

$$I_h(x_0) = [x_0 - h, x_0 + h] \tag{17}$$

where $h \le \min\left\{\delta, \frac{e}{K}\right\}$. Then, there is a unique function $x \to y(x)$, defined for x in $I_h(x_0)$, with continuous first and second derivatives, such that the BVP Equation (1) holds.

The implication of Theorem 3.1 is that it helps us make predictions on the length of the interval on which the solution exists for the BVP Equation (1). More Theorems on the conditions for uniqueness and existence of solutions of two-point BVPs of the Equations (1) and (2) can be found in the book of Agarwal (1989).

4. Results and Discussion

The RADM developed shall be applied on some special linear and nonlinear two-point BVPs of the Equations (1) and (2). This is aimed at testing the computational reliability of the RADM. The following notations shall be used in the tables below

y_{Exact} - Exact/analytical solution

y_{RADM} - Numerical solution using the RADM

4.1. Problem 1

Consider the linear ninth-order two-point BVP

$$y^{(ix)}(x) = -9e^x + y(x), \quad 0 < x < 1 \tag{18}$$

subject to the boundary conditions,

$$\left.\begin{aligned} y^{(j)}(0) &= (1-j),\ j = 0,1,2,3,4 \\ y^{(j)}(1) &= -je,\ j = 0,1,2,3 \end{aligned}\right\} \tag{19}$$

The analytical/exact solution of Equation (18) is

$$y(x) = (1-x)e^x \tag{20}$$

We write Equation. (18) in an operator form as,

$$Ly = -9e^x + y(x) \tag{21}$$

where the differential operator L is given by

$$L = \frac{d^9}{dx^9} \tag{22}$$

Applying L^{-1}, a nine-fold inverse integral operator

$$L^{-1}(\bullet) = \int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x (\bullet)\,dxdxdxdxdx\ \ dxdxdxdx \tag{23}$$

on both sides of Equation (21), we obtain

$$L^{-1}Ly=-9L^{-1}\left(e^{x}\right)+L^{-1}\left(y(x)\right) \tag{24}$$

$$\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x \left(\frac{d^9}{dx^9}\right)dxdxdxdxdx\ \ dxdxdxdx\ =-9L^{-1}\left(e^{x}\right)+L^{-1}\left(y(x)\right)$$

$$\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x \left(\frac{d^8y}{dx^8}-\frac{d^8y(0)}{dx^8}\right)dxdxdxdxdx\ \ dxdxdx\ =-9L^{-1}\left(e^{x}\right)+L^{-1}\left(y(x)\right)$$

$$\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x \left(\frac{d^7y}{dx^7}-\frac{d^7y(0)}{dx^7}-x\frac{d^8y(0)}{dx^8}\right)dxdxdxdxdx\ \ dxdx\ =-9L^{-1}\left(e^{x}\right)+L^{-1}\left(y(x)\right)$$

$$\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x \left(\frac{d^6y}{dx^6}-\frac{d^6y(0)}{dx^6}-x\frac{d^7y(0)}{dx^7}-\frac{x^2}{2}\frac{d^8y(0)}{dx^8}\right)dxdxdxdxdx\ \ dx$$
$$=-9L^{-1}\left(e^{x}\right)+L^{-1}\left(y(x)\right)$$

$$\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x \left(\frac{d^5y}{dx^5}-\frac{d^5y(0)}{dx^5}-x\frac{d^6y(0)}{dx^6}-\frac{x^2}{2}\frac{d^7y(0)}{dx^7}-\frac{x^3}{6}\frac{d^8y(0)}{dx^8}\right)dxdxdxdxdx$$
$$=-9L^{-1}\left(e^{x}\right)+L^{-1}\left(y(x)\right)$$

$$\int_0^x\int_0^x\int_0^x\int_0^x \left(\frac{d^4y}{dx^4}-\frac{d^4y(0)}{dx^4}-x\frac{d^5y(0)}{dx^5}-\frac{x^2}{2}\frac{d^6y(0)}{dx^6}-\frac{x^3}{6}\frac{d^7y(0)}{dx^7}-\frac{x^4}{24}\frac{d^8y(0)}{dx^8}\right)dxdxdxdx$$
$$=-9L^{-1}\left(e^{x}\right)+L^{-1}\left(y(x)\right)$$

$$\int_0^x\int_0^x\int_0^x \left(\begin{array}{l}\frac{d^3y}{dx^3}-\frac{d^3y(0)}{dx^3}-x\frac{d^4y(0)}{dx^4}-\frac{x^2}{2}\frac{d^5y(0)}{dx^5}\\ -\frac{x^3}{6}\frac{d^6y(0)}{dx^6}-\frac{x^4}{24}\frac{d^7y(0)}{dx^7}-\frac{x^5}{120}\frac{d^8y(0)}{dx^8}\end{array}\right)dxdxdx\ =-9L^{-1}\left(e^{x}\right)+L^{-1}\left(y(x)\right)$$

$$\int_0^x\int_0^x \left(\begin{array}{l}\frac{d^2y}{dx^2}-\frac{d^2y(0)}{dx^2}-x\frac{d^3y(0)}{dx^3}-\frac{x^2}{2}\frac{d^4y(0)}{dx^4}-\frac{x^3}{6}\frac{d^5y(0)}{dx^5}\\ -\frac{x^4}{24}\frac{d^6y(0)}{dx^6}-\frac{x^5}{120}\frac{d^7y(0)}{dx^7}-\frac{x^6}{720}\frac{d^8y(0)}{dx^8}\end{array}\right)dxdx\ =-9L^{-1}\left(e^{x}\right)+L^{-1}\left(y(x)\right)$$

$$\int_0^x \left(\begin{array}{l}\frac{dy}{dx}-\frac{dy(0)}{dx}-x\frac{d^2y(0)}{dx^2}-\frac{x^2}{2}\frac{d^3y(0)}{dx^3}-\frac{x^3}{6}\frac{d^4y(0)}{dx^4}\\ -\frac{x^4}{24}\frac{d^5y(0)}{dx^5}-\frac{x^5}{120}\frac{d^6y(0)}{dx^6}-\frac{x^6}{720}\frac{d^7y(0)}{dx^7}-\frac{x^7}{5040}\frac{d^8y(0)}{dx^8}\end{array}\right)=-9L^{-1}\left(e^{x}\right)+L^{-1}\left(y(x)\right)$$

$$y(x)-y(0)-x\frac{dy(0)}{dx}-\frac{x^2}{2}\frac{d^2y(0)}{dx^2}-\frac{x^3}{6}\frac{d^3y(0)}{dx^3}-\frac{x^4}{24}\frac{d^4y(0)}{dx^4}-\frac{x^5}{120}\frac{d^5y(0)}{dx^5}$$
$$-\frac{x^6}{720}\frac{d^6y(0)}{dx^6}-\frac{x^7}{5040}\frac{d^7y(0)}{dx^7}-\frac{x^8}{40320}\frac{d^8y(0)}{dx^8}=-9L^{-1}(e^{x})+L^{-1}\left(y(x)\right) \tag{25}$$

Equation (25) reduces to

$$y(x)-\varphi_1-\varphi_2x-\varphi_3\frac{x^2}{2!}-\varphi_4\frac{x^3}{3!}-\varphi_5\frac{x^4}{4!}-\varphi_6\frac{x^5}{5!}$$
$$-\varphi_7\frac{x^6}{6!}-\varphi_8\frac{x^7}{7!}-\varphi_9\frac{x^8}{8!}=-9L^{-1}(e^{x})+L^{-1}\left(y(x)\right) \tag{26}$$

where

$$\varphi_1=y(0),\varphi_2=\frac{dy(0)}{dx},\varphi_3=\frac{dy^2(0)}{dx^2},\ldots,\varphi_8=\frac{dy^8(0)}{dx^8}$$

From the boundary condition given in Equation (19), it is clear that

$$\varphi_1=1,\varphi_2=0,\varphi_3=-1,\varphi_4=-2,\varphi_5=-3$$

and let us assume that

$$\varphi_6=\alpha,\varphi_7=\beta,\varphi_8=\gamma\ \ and\ \ \varphi_9=\lambda$$

then Equation. (26) becomes

$$y(x)=1-\frac{x^2}{2!}-\frac{x^3}{3}-\frac{x^4}{8}+\alpha\frac{x^5}{5!}+\beta\frac{x^6}{6!}+\gamma\frac{x^7}{7!}$$
$$+\lambda\frac{x^8}{8!}-9L^{-1}(e^{x})+L^{-1}\left(y(x)\right) \tag{27}$$

Recall that the ADM expresses the solution $y(x)$ to the BVP in Equation. (18) by the decomposition series in Equation. (8). This implies that Equation (27) becomes

$$\sum_{n=0}^{\infty}y(x)=1-\frac{x^2}{2!}-\frac{x^3}{3}-\frac{x^4}{8}+\alpha\frac{x^5}{5!}+\beta\frac{x^6}{6!}+\gamma\frac{x^7}{7!}$$
$$+\lambda\frac{x^8}{8!}-9L^{-1}(e^{x})+L^{-1}\left(\sum_{n=0}^{\infty}y_n(x)\right) \tag{28}$$

The algorithm for obtaining y is, therefore, given by

$$\left.\begin{array}{l}y_0(x)=1\\ y_1(x)=-\frac{x^2}{2!}-\frac{x^3}{3}-\frac{x^4}{8}+\alpha\frac{x^5}{5!}+\beta\frac{x^6}{6!}+\gamma\frac{x^7}{7!}+\lambda\frac{x^8}{8!}-9L^{-1}(e^{x})+L^{-1}\left(\sum_{n=0}^{\infty}y_n(x)\right)\\ y_{n+1}(x)=L^{-1}\left(y_n(x)\right),\ n\geq 1\end{array}\right\} \tag{29}$$

Note that

$$L^{-1}(e^{x})=\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x e^{x}\,dxdxdxdxdx\ \ dxdxdxdx$$
$$=e^{x}-1-x-\frac{x^2}{2!}-\frac{x^3}{3!}-\ldots-\frac{x^8}{8!}$$

Thus,

$$-9L^{-1}(e^{x})=-\frac{9x^9}{9!}-\frac{9x^{10}}{10!}-\frac{9x^{11}}{11!}-\ldots \tag{30}$$

Also,

$$L^{-1}\left(y_0(x)\right)=\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x dxdxdxdxdx\ \ dxdxdxdx\ =\frac{x^9}{9!} \tag{31}$$

Therefore, adding Equations (30) and (31), we obtain

$$-9L^{-1}(e^{x})+L^{-1}\left(y_0(x)\right)=-\frac{x^9}{45360}-\frac{x^{10}}{403200}-\frac{x^{11}}{443500}-\ldots \tag{32}$$

The components of (25) is therefore determined as follows [by substituting Eq. (32) in Eq. (29)],

$$\left.\begin{array}{l} y_0(x)=1 \\ y_1(x)=-\frac{x^2}{2!}-\frac{x^3}{3}-\frac{x^4}{8}+\alpha\frac{x^5}{5!}+\beta\frac{x^6}{6!}+\gamma\frac{x^7}{7!}+\lambda\frac{x^8}{8!}-\frac{x^9}{45360}-\frac{x^{10}}{403200}-\frac{x^{11}}{4435200}-\ldots \\ y_2(x)=L^{-1}\left(y_1(x)\right)=-\frac{x^{11}}{39916800}-\frac{x^{12}}{239500800}+\ldots \\ y_3(x)=L^{-1}\left(y_2(x)\right)=\frac{x^{13}}{13!}+\ldots \end{array}\right\} \tag{33}$$

Approximating the solution by using only these four terms y_0, y_1, y_2 *and* y_3, the solution series is given by

$$\begin{aligned} y(x) &= y_0+y_1+y_2+y_3=1-\frac{x^2}{2!}-\frac{x^3}{3}-\frac{x^4}{8}+\alpha\frac{x^5}{5!}+\beta\frac{x^6}{6!}+\gamma\frac{x^7}{7!}+\lambda\frac{x^8}{8!} \\ &\quad -\frac{x^9}{45360}-\frac{x^{10}}{403200}-\frac{x^{11}}{3991680}-\frac{x^{12}}{239500800}+\frac{x^{13}}{13!}+\ldots \end{aligned} \tag{34}$$

In order to determine the values of the constants α, β, γ *and* λ, we use the boundary conditions at $x=1$ and substituting it back in Equation. (34). Therefore,

$$\begin{aligned} y(1) &= 1-\frac{1}{2!}-\frac{1}{3}-\frac{1}{8}+\alpha\frac{1}{5!}+\beta\frac{1}{6!}+\gamma\frac{1}{7!}+\lambda\frac{1}{8!} \\ &\quad -\frac{1}{45360}-\frac{1}{403200}-\frac{1}{3991680}-\frac{1}{239500800}+\frac{1}{13!}+\ldots=0 \end{aligned} \tag{35}$$

$$\begin{aligned} y'(x) &= x-x^2-\frac{x^3}{2}+\alpha\frac{x^4}{24}+\beta\frac{x^5}{120}+\gamma\frac{x^6}{720}+\lambda\frac{x^7}{5040} \\ &\quad -\frac{x^8}{5040}-\frac{x^9}{40320}-\frac{x^{10}}{362880}-\frac{x^{11}}{19958400}+\frac{x^{12}}{479001600}=-e \end{aligned} \tag{36}$$

$$\begin{aligned} y''(x) &= 1-2x-\frac{3x^2}{2}+\alpha\frac{x^3}{6}+\beta\frac{x^4}{24}+\gamma\frac{x^5}{120}+\lambda\frac{x^6}{720} \\ &\quad -\frac{x^7}{630}-\frac{x^8}{4480}-\frac{x^9}{36288}-\frac{x^{10}}{1814400}+\frac{x^{11}}{39916800}=-2e \end{aligned} \tag{37}$$

$$\begin{aligned} y'''(x) &= -2-6x+\alpha\frac{x^2}{2}+\beta\frac{x^3}{8}+\gamma\frac{x^4}{24}+\lambda\frac{x^5}{120} \\ &\quad -\frac{x^6}{90}-\frac{x^7}{560}-\frac{x^8}{4032}-\frac{x^9}{181440}+\frac{x^{10}}{362880}=-3e \end{aligned} \tag{38}$$

We also compute Equations. (36–38) at $x=1$. Solving the equations, we obtain the values of the constants

$$\alpha=-4.0000,\ \beta=-5.0002,\ \gamma=-5.9985\ \ and \ \ \lambda=-7.0050 \tag{39}$$

Thus, the series solution to Equation. (18) in Problem 1 is given by,

$$\begin{aligned} y(x) &\approx 1-\frac{x^2}{2!}-\frac{x^3}{3}-\frac{x^4}{8}-0.033333333x^5-0.006944722x^6-0.001190179x^7 \\ &\quad -0.000173735x^8-0.000022046x^9-0.000002480x^{0}-0.000000251x^{1}+\ldots \end{aligned} \tag{40}$$

The result presented in Table 1 and Figure 1.

Table 1. Results of Problem 1 using the RADM.

x	y_{Exact}	y_{RADM}	Absolute error
0.0000	1.000000000	1.000000000	0.0000 e+000
0.1000	0.994653826	0.994653826	2.1125 e−009
0.2000	0.977122207	0.977122207	2.9134 e−009
0.3000	0.944901165	0.944901165	3.0010 e−009
0.4000	0.895094819	0.895094819	3.2001 e−009
0.5000	0.824360635	0.824360635	1.0124 e−010
0.6000	0.728847520	0.728847520	1.4567 e−010
0.7000	0.604125812	0.604125812	2.0192 e−010
0.8000	0.445108186	0.445108186	2.2198 e−010
0.9000	0.245960311	0.245960311	3.1017 e−010
1.0000	0.000000000	0.000000000	0.0000 e+000

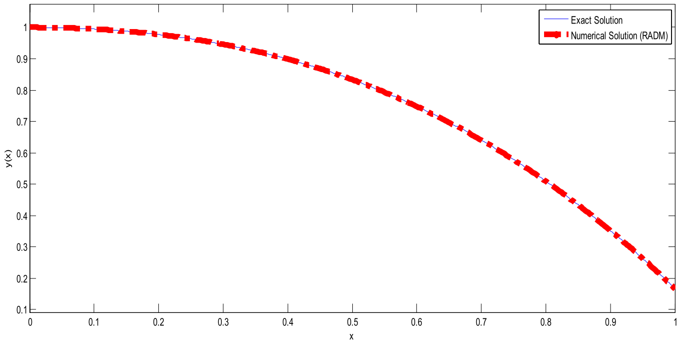

Figure 1. Solution curves for Problem 1.

4.2. Problem 2

Consider the nonlinear sixth-order two-point BVP

$$y^{(vi)}(x)=e^{-x}y^2(x),\quad 0<x<1 \tag{41}$$

subject to the boundary conditions,

$$\left.\begin{array}{l} y^{(2j)}(0)=1,\ j=0,1,2 \\ y^{(2j)}(1)=e,\ j=0,1,2 \end{array}\right\} \tag{42}$$

The analytical/exact solution of Equation (41) is

$$y(x)=e^x \tag{43}$$

We write Equation. (41) in an operator form as,

$$Ly=e^{-x}Ny=e^{-x}y^2(x) \tag{44}$$

where the differential operator L is given by,

$$L=\frac{d^6}{dx^6} \tag{45}$$

and Ny is the nonlinear term.

Applying L^{-1}, a six-fold inverse integral operator

$$L^{-1}(\bullet)=\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x(\bullet)dxdxdxdxdx\ \ dx \qquad (46)$$

on both sides of Equation (44), we obtain

$$L^{-1}Ly=L^{-1}\left(e^{-x}y^2(x)\right) \qquad (47)$$

$$\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\left(\frac{d^6y}{dx^6}\right)dxdxdxdxdx\ \ dx=L^{-1}\left(e^{-x}y^2(x)\right)$$

$$\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\left(\frac{d^5y}{dx^5}-\frac{d^5y(0)}{dx^5}\right)dxdxdxdxdx\ \ =L^{-1}\left(e^{-x}y^2(x)\right)$$

$$\int_0^x\int_0^x\int_0^x\int_0^x\left(\frac{d^4y}{dx^4}-\frac{d^4y(0)}{dx^4}-x\frac{d^5y(0)}{dx^5}\right)dxdxdxdx\ \ =L^{-1}\left(e^{-x}y^2(x)\right)$$

$$\int_0^x\int_0^x\int_0^x\left(\frac{d^3y}{dx^3}-\frac{d^3y(0)}{dx^3}-x\frac{d^4y(0)}{dx^4}-\frac{x^2}{2}\frac{d^5y(0)}{dx^5}\right)dxdxdx\ \ =L^{-1}\left(e^{-x}y^2(x)\right)$$

$$\int_0^x\int_0^x\left(\frac{d^2y}{dx^2}-\frac{d^2y(0)}{dx^2}-x\frac{d^3y(0)}{dx^3}-\frac{x^2}{2}\frac{d^4y(0)}{dx^4}-\frac{x^3}{6}\frac{d^5y(0)}{dx^5}\right)dxdx\ \ =L^{-1}\left(e^{-x}y^2(x)\right)$$

$$\int_0^x\left(\frac{dy}{dx}-\frac{dy(0)}{dx}-x\frac{d^2y(0)}{dx^2}-\frac{x^2}{2}\frac{d^3y(0)}{dx^3}-\frac{x^3}{6}\frac{d^4y(0)}{dx^4}-\frac{x^4}{24}\frac{d^5y(0)}{dx^5}\right)=L^{-1}\left(e^{-x}y^2(x)\right)$$

$$y(x)-y(0)-x\frac{dy(0)}{dx}-\frac{x^2}{2}\frac{d^2y(0)}{dx^2}-\frac{x^3}{6}\frac{d^3y(0)}{dx^3}-\frac{x^4}{24}\frac{d^4y(0)}{dx^4}-\frac{x^5}{120}\frac{d^5y(0)}{dx^5}$$

$$=L^{-1}\left(e^{-x}y^2(x)\right) \qquad (48)$$

Equation (48) reduces to

$$y(x)-\varphi_1-\varphi_2x-\varphi_3\frac{x^2}{2!}-\varphi_4\frac{x^3}{3!}-\varphi_5\frac{x^4}{4!}-\varphi_6\frac{x^5}{5!}=L^{-1}\left(e^{-x}y^2(x)\right) \qquad (49)$$

where

$$\varphi_1=y(0),\varphi_2=\frac{dy(0)}{dx},\varphi_3=\frac{dy^2(0)}{dx^2},\ldots,\varphi_6=\frac{dy^5(0)}{dx^5}$$

From the boundary condition given in Equation (42), it is clear that

$$\varphi_1=\varphi_3=\varphi_5=1$$

and let us assume that the unknown constants

$$\varphi_2=\alpha,\varphi_4=\beta\ and\ \ \varphi_6=\gamma$$

then Equation. (49) becomes

$$y(x)=1+\alpha x+\frac{x^2}{2}+\frac{\beta x^3}{3}+\frac{x^4}{24}+\frac{\gamma x^5}{120}+L^{-1}\left[e^{-x}y^2(x)\right] \qquad (50)$$

Recall that the ADM expresses the solution $y(x)$ to the BVP in Equation. (41) by the decomposition series in Equation. (8). This implies that Equation (50) becomes

$$\sum_{n=0}^{\infty}y(x)=1+\alpha x+\frac{x^2}{2}+\frac{\beta x^3}{6}+\frac{x^4}{24}+\frac{\gamma x^5}{120}+L^{-1}\left[e^{-x}\sum_{n=0}^{\infty}A_n\right] \qquad (51)$$

The algorithm for obtaining y is therefore given by

$$\left.\begin{aligned}&y_0(x)=1\\&y_1(x)=\alpha x+\frac{x^2}{2}+\frac{\beta x^3}{6}+\frac{x^4}{24}+\frac{\gamma x^5}{120}+L^{-1}\left[e^{-x}A_0\right]\\&y_{n+1}(x)=L^{-1}\left(e^{-x}A_n\right),\ n\geq 1\end{aligned}\right\} \qquad (52)$$

We determine the components recursively as follows,

$$A_0=N(y_0)=y_0^2=1 \qquad (53)$$

$$L^{-1}\left[e^{-x}A_0\right]=L^{-1}\left[e^{-x}\right]=\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x e^{-x}dxdxdxdxdx\ \ dx$$

$$=e^{-x}+\frac{x^5}{120}-\frac{x^4}{24}+\frac{x^3}{6}-\frac{x^2}{2}+x-1 \qquad (54)$$

Substituting Equation (54) in y_1 in Equation. (52), we get

$$y_1=-1+(\alpha+1)x+\left(\frac{\beta}{6}+\frac{1}{6}\right)x^3+\left(\frac{\gamma}{120}+\frac{1}{120}\right)x^5+e^{-x} \qquad (55)$$

Approximating the solution by using only these two terms $y_0\ and\ y_1$, the solution series is given by

$$y(x)=y_0+y_1=(\alpha+1)x+\left(\frac{\beta}{6}+\frac{1}{6}\right)x^3+\left(\frac{\gamma}{120}+\frac{1}{120}\right)x^5+e^{-x} \qquad (56)$$

We use Taylor series expansion of e^{-x}, that is $1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\ldots$ in Equation. (56) to obtain the final approximation formula of $y(x)$ as

$$y(x)\approx 1+\alpha x+\frac{x^2}{2}+\frac{\beta x^3}{6}+\frac{x^4}{24}+\frac{\gamma x^5}{120}+\ldots \qquad (57)$$

In order to determine the values of the constants $\alpha,\beta\ and\ \gamma$, we use the boundary conditions at $x=1$ and substituting it back in Equation. (57). Therefore,

$$y(1)=1+\alpha+\frac{1}{2}+\frac{\beta}{6}+\frac{1}{24}+\frac{\gamma}{120}+\ldots=e \qquad (58)$$

$$y'(x)=\alpha+x+\beta\frac{x^2}{2}+\frac{x^3}{6}+\gamma\frac{x^4}{24}+\ldots=e$$

$$y''(x) = 1 + \beta x + \frac{x^2}{2} + \gamma\frac{x^3}{6} + \ldots = e$$

$$\Rightarrow\ y''(1) = 1 + \beta + \frac{1}{2} + \frac{\gamma}{6} + \ldots = e \qquad (59)$$

$$y'''(x) = \beta + x + \gamma\frac{x^2}{2} + \ldots = e$$

$$y^4(x) = 1 + \gamma x + \ldots \Rightarrow\ y^4(1) = 1 + \gamma + \ldots = e \qquad (60)$$

From (60),

$$\gamma = e - 1 = 1.718281828 \qquad (61)$$

Then from (59), we find β

$$\beta = e - 1 - \frac{1}{2} - \frac{1.71828183}{6} = 0.931901523 \qquad (62)$$

Substituting Equations (61) and (62) in Equation (58), we obtain α as

$$\alpha = e - \frac{1.71828183}{120} - 1 - 0.5 - \frac{0.93190153}{6} = 1.006979226 \qquad (63)$$

Thus, the series solution to Problem 2 in Equation. (41) is given by

$$y(x) \approx 1 + 1.006979226\ \ x + \frac{x^2}{2} + 0.9319015233\ \ \frac{x^3}{6} + \frac{x^4}{24} + 1.718281828\ \ \frac{x^5}{120} + \ldots \qquad (64)$$

The result is presented in Table 2 and Figure 2.

Table 2. Results of Problem 2 using the RADM.

x	y_{Exact}	y_{RADM}	Absolute error
0.0000	1.000000000	1.000000000	0.0000 e+000
0.1000	1.105170918	1.105101984	6.8934 e−005
0.2000	1.221402758	1.221481981	7.9223 e−005
0.3000	1.349858808	1.349800131	5.8677 e−006
0.4000	1.491824698	1.491822112	2.5860 e−006
0.5000	1.648721271	1.648714131	2.7141 e−006
0.6000	1.822118800	1.822121002	2.2020 e−006
0.7000	2.013752707	2.013750349	2.3580 e−006
0.8000	2.225540928	2.225540319	6.0900 e−007
0.9000	2.459603111	2.459601903	1.2080 e−007
1.0000	2.718281828	2.718281984	1.5600 e−007

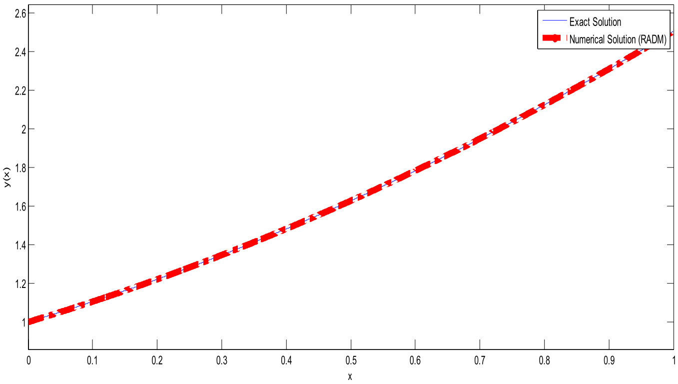

Figure 2. Solution curves for Problem 2.

4.3. Problem 3

Consider the linear seventh-order two-point BVP

$$y^{(vii)}(x) = e^x(x^2 - 2x - 6) + xy(x),\quad 0 < x < 1 \qquad (65)$$

subject to the boundary conditions,

$$\left.\begin{array}{l} y^{(j)}(0) = (1 - j),\ j = 0,1,2,3 \\ y^{(j)}(1) = -je,\ j = 0,1,2 \end{array}\right\} \qquad (66)$$

The analytical/exact solution of the problem Equation (65) is

$$y(x) = e^x - xe^x \qquad (67)$$

We write Equation. (65) in an operator form as

$$Ly = e^x(x^2 - 2x - 6) + xy(x) \qquad (68)$$

where the differential operator L is given by

$$L = \frac{d^7}{dx^7} \qquad (69)$$

Applying L^{-1}, a seventh-fold inverse integral operator

$$L^{-1}(\bullet) = \int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x\int_0^x(\bullet)\,dxdxdxdxdx\ \ dxdx \qquad (70)$$

on both sides of Equation (68) and imposing the boundary conditions in Equation (2) at the point $x = 1$ by using the procedure adopted in Problem 1, we obtain the solution series

$$y(x) = -63 - 64x - \frac{35}{2}x^2 - 4x^3 + \left(\frac{\alpha}{24} - \frac{1}{2}\right)x^4 + \left(\frac{\beta}{120} - \frac{1}{30}\right)\beta x^5 + \frac{1}{360}(2 + \gamma)x^6 + e^x(x - 8)^2 + L^{-1}(xy(x)) \qquad (71)$$

where α, β *and* γ are constants and

$$\left.\begin{aligned} y^{(4)}(0) &= \alpha \\ y^{(5)}(0) &= \beta \\ y^{(6)}(0) &= \gamma \end{aligned}\right\} \tag{72}$$

Recall that from the ADM, the solution $y(x)$ to the BVP in Equation. (65) can be expressed by Equation. (8). This implies that Equation (71) becomes,

$$\sum_{n=0}^{\infty} y(x) = -63 - 64x - \frac{35}{2}x^2 - 4x^3 + \left(\frac{\alpha}{24} - \frac{1}{2}\right)x^4 + \left(\frac{\beta}{120} - \frac{1}{30}\right)\beta x^5 + \frac{1}{360}(2+\gamma)x^6 + e^x(x-8)^2 + L^{-1}\left(x\sum_{n=0}^{\infty} y_n(x)\right) \tag{73}$$

The algorithm for obtaining y is thus given by the recursive relation

$$\left.\begin{aligned} y_0(x) &= -63 - 64x - \frac{35}{2}x^2 - 4x^3 + \left(\frac{\alpha}{24} - \frac{1}{2}\right)x^4 + \left(\frac{\beta}{120} - \frac{1}{30}\right)\beta x^5 \\ &\quad + \frac{1}{360}(2+\gamma)x^6 + e^x(x-8)^2 \\ y_{n+1}(x) &= L^{-1}(xy_n(x)),\ n \geq 0 \end{aligned}\right\} \tag{74}$$

In order to determine the values of the constants α, β *and* γ, we use the boundary conditions at $x = 1$ on the first four terms given by,

$$y = y_0 + y_1 + y_2 + y_3 \tag{75}$$

Solving the resulting equations, we obtain the values of α, β *and* γ as

$$\alpha = -3.000000\ ,\ \beta = -3.999999 \ \ and \ \ \gamma = -5.000002 \tag{76}$$

Therefore, the series solution to Equation. (65) in Problem 3 is given by,

$$y(x) \approx 1 - \frac{x^2}{2} - \frac{x^3}{3} - 0.12500\ x^4 - 0.03333\ x^5 - 0.00694\ x^6 - \frac{x^7}{840} - \frac{x^8}{5760} - \frac{x^9}{45360} - \frac{x^{10}}{403200} - \frac{x^{11}}{3991680} - \frac{287\ x^{12}}{1250000000\ 0} - \frac{803\ x^{13}}{5000000000\ 000} + \ldots \tag{77}$$

The result presented in Table 3 and Figure 3.

Table 3. Results of Problem 3 using the RADM.

x	y_{Exact}	y_{RADM}	Absolute error
0.0000	1.000000000	1.000000000	0.0000 e+000
0.1000	0.994653826	0.994653819	7.0100 e−008
0.2000	0.977122206	0.977122221	1.5000 e−008
0.3000	0.944901166	0.944901178	1.2000 e−008
0.4000	0.895094819	0.895094829	1.1101 e−008
0.5000	0.824360636	0.824360611	2.5000 e−008
0.6000	0.728847520	0.728847514	6.0000 e−008
0.7000	0.604125812	0.604125841	2.9000 e−008
0.8000	0.445108186	0.445108136	5.0000 e−008
0.9000	0.245960311	0.245960347	3.6964 e−008
1.0000	0.000000000	0.000000012	1.2000 e−008

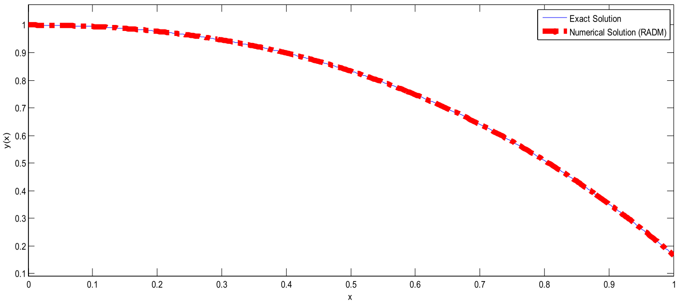

Figure 3. Solution curves for Problem 3.

5. Discussion of Results

The RADM derived in this work has been applied on some special linear and nonlinear two-point BVPs. The method generated approximate series solution to the BVPs of the Equation (1). It was observed that the numerical and graphical solutions of the RADM converge rapidly to the exact solution of the BVPs even with fewer terms. This implies that the RADM is computationally efficient.

6. Conclusion

A RADM for the approximation of special linear and nonlinear two-point BVP has been derived in this research. It was observed that the method was applied without the need for any form of linearization, discretization, perturbation or even transformation. From the numerical and graphical results obtained, it is clear that the RADM converges rapidly to the analytical/exact solution even with fewer numbers of terms.

Conflict of interest

The authors declare no conflict of interests.

References

Adomian G. Solving frontier problems of physics: the decomposition method, Kluwer, MA, Boston, MA, 1994.

Agarwal PP. Boundary value problems for higher ordinary differential equations. World Scientific, Singapore, 1989.

Akram G, Aslam IA. Solution of fourth order three-point boundary value problems using Aadomian decomposition method and Runge-Kutta method. J Assoc Arab Univ Basic Appl Sci 2018; 20(1):61–7.

Butcher JC. Numerical methods for ODEs. 2nd edition, John Wiley & Sons Ltd, Chichester, UK, 2008.

Duan J, Rach R, Wazwaz AM, Chaolu T, Wang Z. A new modified Aadomian decomposition method and its multistage form for solving nonlinear boundary value problems with Robin boundary conditions. Appl Math Model 2013; 37:8687–708; doi:10.1016/j.apm.2013.02.002

Henrici P. Discrete variable methods in ordinary differential equations. John Wiley and Sons, New York, 1962.

Ibijola EA, Sunday J. On the computational procedure of solving boundary value problems of class of M using power series method. Aust J Basic Appl Sci 2010; 4(6):1007–14.

Jang B. Two-point boundary value problems by the extended Aadomian decomposition method. J Comput Appl Math 2008; 219:253–63; doi:10.1016/j.cam.2007.07.036

Javed A, Islam S, Sirajul I, Gul Z. The solution of multi-point boundary value problems by the optimal homotopy asymptotic method. Comput Math Appl 2010; 59(6):2000–6; doi:10.1016/j.camwa.2009.12.002.

Mohyud-Din ST, Noor MA, Noor KI. Exp-function method for solving higher-order boundary value problems. Bull Inst Math Acad Sin 2009; 4(2):219–34.

Nadirah MN, Zanariah AM, Fudziah I, Norfifah B. Diagonal block method for solving two-point boundary value problems with Robin boundary conditions. Math Probl Eng 2018; 2018:1–12.

Opanuga AA, Okagbue HI, Owoloko EA, Agboola OO. Modified Aadomian decomposition method for thirteenth-order boundary value problems. Gazi Univ J Sci 2017; 30(4):454–61.

Phang PS, Zanariah AM, Mohammed S. Solving nonlinear two-point boundary value problem using two-step direct method. J Qual Meas Anal 2017; 7(1):129–40.

Rashidinia J, Ghasemi M. B-spline collocation for solution of two-point boundary value problems. J Comput Appl Math 2011; 235(8):2325–42; doi:10.1016/j.cam.2010.10.031.

Sivakumar TR, Baiju S. Shooting type Laplace Adomian decomposition algorithm for nonlinear differential equations with boundary conditions at infinity. Appl Math Lett 2011; 24(10):1702–8; doi:10.1016/j.aml.2011.04.024.

Waleed A. Adomian decomposition method with Green's function for sixth-order boundary value problems. Comput Math Appl 2011; 61(6):1567–75; doi:10.1016/j.camwa.2011.01.025.

Wazwaz AM. Approximate solutions to boundary value problems of higher order by the modified decomposition method. Comput Math Appl 2000; 40:679–91; doi:10.1016/S0898-1221(00)00187-5.