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Velocity and Acceleration in Oblate Spheroidal Coordinates

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ABSTRACT

The expressions for instantaneous velocity and acceleration in Cartesian, cylindrical and spherical coordinates and their applications in mechanics are well known. It is, however, now well known [1-6] that the planets and sun and all rotating astronomical bodies are more precisely spheroidal in geometry. And their fields as well as the motions of test particles in them require the use of spheroidal coordinates. Consequently in this paper we derive the expressions for instantaneous velocity and acceleration in oblate spheroidal coordinates for application to the motions of test particles in the fields of the spheroidal astronomical bodies.

INTRODUTION

The oblate spheroidal coordinates (η, ξ, ϕ) are defined in terms of the Cartesian coordinates (x, y, z) by

$$x = a\left(1-\eta^2\right)^{\frac{1}{2}}\left(1+\xi^2\right)^{\frac{1}{2}}\cos\phi \tag{1}$$

$$y = a\left(1-\eta^2\right)^{\frac{1}{2}}\left(1+\xi^2\right)^{\frac{1}{2}}\sin\phi \tag{2}$$

$$z = a\,\eta\,\xi \tag{3}$$

where a is a constant parameter

$$-1 \le \eta \le 1; \quad o \le \xi \le \infty; \quad o \le \phi \le 2\pi \tag{4}$$

Consequently, by definition, the oblate spheroidal scale factors or metrical coefficients are given by

$$h_\eta = a\left(\frac{\eta^2+\xi^2}{1-\eta^2}\right)^{\frac{1}{2}} \tag{5}$$

$$h_\xi = a\left(\frac{\eta^2+\xi^2}{1+\xi^2}\right)^{\frac{1}{2}} \tag{6}$$

$$h_\phi = a\left(1-\eta^2\right)^{\frac{1}{2}}\left(1+\xi^2\right)^{\frac{1}{2}} \tag{7}$$

These scale factors define the unit vectors, line element, volume element, as well as gradient, divergence, curl and laplacian operators in oblate spheroidal coordinates, according to the theory of orthogonal curvilinear coordinates[7-11]. And these quantities are necessary and sufficient for the derivation of the fields of all oblate spheroidal distributions of mass and charge and current. Now for the formulation of the equations of motion for test particles in these fields we shall investigate the expression for instantaneous velocity and acceleration in oblate spheroidal coordinates.

MATHEMATICAL ANALYSIS

By definition the oblate spheroidal unit vectors are given in terms of the Cartesian unit vectors as

$$\hat{\eta} = -\frac{\eta\left(1+\xi^2\right)^{\frac{1}{2}}\cos\phi}{\left(\eta^2+\xi^2\right)^{\frac{1}{2}}}\hat{\imath} - \frac{\eta\left(1+\xi^2\right)^{\frac{1}{2}}\sin\phi}{\left(\eta^2+\xi^2\right)^{\frac{1}{2}}}\hat{\jmath} + \frac{\xi\left(1-\eta^2\right)^{\frac{1}{2}}}{\left(\eta^2+\xi^2\right)^{\frac{1}{2}}}\hat{\kappa} \tag{8}$$

$$\hat{\xi} = \frac{\xi(1-\eta^2)^{\frac{1}{2}}\cos\phi}{(\eta^2+\xi^2)^{\frac{1}{2}}}\hat{\imath} + \frac{\xi(1-\eta^2)^{\frac{1}{2}}\sin\phi}{(\eta^2+\xi^2)^{\frac{1}{2}}}\hat{\jmath} + \frac{\eta(1+\xi^2)}{(\eta^2+\xi^2)^{\frac{1}{2}}}\hat{\kappa} \qquad (9)$$

$$\hat{\phi} = -\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath} \qquad (10)$$

Hence denoting one time differentiation by a dot, it follows from (8) and (9), (10) and some manipulation that

$$\dot{\hat{\eta}} = \frac{1}{(\eta^2+\xi^2)}\left\{-\left(\frac{1+\xi^2}{1-\eta^2}\right)^{\frac{1}{2}}\xi\dot{\eta} + \left(\frac{1-\eta^2}{1+\xi^2}\right)^{\frac{1}{2}}\eta\dot{\xi}\right\}\hat{\xi} - \left(\frac{1+\xi^2}{\eta^2+\xi^2}\right)^{\frac{1}{2}}\eta\,\dot{\phi}\,\hat{\phi} \qquad (11)$$

Similarly, it follows from (9) and (8), (10) that

$$\dot{\hat{\xi}} = \frac{1}{(\eta^2+\xi^2)}\left\{\left(\frac{1+\xi^2}{1-\eta^2}\right)^{\frac{1}{2}}\xi\dot{\eta} - \left(\frac{1-\eta^2}{1+\xi^2}\right)^{\frac{1}{2}}\eta\dot{\xi}\right\}\hat{\eta}$$

$$+\left(\frac{1-\eta^2}{\eta^2+\xi^2}\right)^{\frac{1}{2}}\xi\,\dot{\phi}\,\hat{\phi} \qquad (12)$$

Similarly, it follows from (10) and (8), (9) that

$$\dot{\hat{\phi}} = \left(\frac{1+\xi^2}{\eta^2+\xi^2}\right)^{\frac{1}{2}}\eta\,\dot{\phi}\,\hat{\eta} - \left(\frac{1-\eta^2}{\eta^2+\xi^2}\right)^{\frac{1}{2}}\xi\,\dot{\phi}\,\hat{\xi} \qquad (13)$$

Now it follows from definition of instantaneous position vector $\underline{r}$, as

$$\underline{r} = x\,\hat{\imath} + y\,\hat{\jmath} + z\,\hat{k} \qquad (14)$$

and (8) - (10) that the instantaneous position vector may be expressed entirely in terms of oblate spheroidal coordinates as

$$\underline{r} = -\frac{a\eta(1-\eta^2)^{\frac{1}{2}}}{(\eta^2+\xi^2)^{\frac{1}{2}}}\hat{\eta} + \frac{a\xi(1+\xi^2)^{\frac{1}{2}}}{(\eta^2+\xi^2)^{\frac{1}{2}}}\hat{\xi} \qquad (15)$$

It now follows from definition of instantaneous velocity vector, $\underline{u}$, as

$$\underline{u} = \dot{\underline{r}} \qquad (16)$$

and (15) and (11) – (13) that the instantaneous velocity vector may be expressed entirely in terms of oblate spheroidal coordinates as

$$\underline{u} = u_\eta \hat{\eta} + u_\xi \hat{\xi} + u_\phi \hat{\phi} \tag{17}$$

where

$$u_\eta = a\left(\frac{\eta^2 + \xi^2}{1-\eta^2}\right)^{\frac{1}{2}} \dot{\eta} \tag{18}$$

$$u_\xi = a\left(\frac{\eta^2 + \xi^2}{1+\xi^2}\right)^{\frac{1}{2}} \dot{\zeta} \tag{19}$$

$$u_\phi = a\left(1-\eta^2\right)^{\frac{1}{2}}\left(1+\xi^2\right)^{\frac{1}{2}} \dot{\phi} \tag{20}$$

Similarly, it follows from definition of instantaneous acceleration, $\underline{a}$,

as

$$a = \dot{\underline{u}} \tag{21}$$

and (18) – (20) and (11) – (13) that the instantaneous acceleration may be expressed entirely in terms of oblate spheroidal coordinates as

$$\underline{a} = a_\eta \hat{\eta} + a_\xi \hat{\xi} + a_\phi \hat{\phi} \tag{22}$$

where

$$a_\eta = \frac{a(\eta^2+\xi^2)^{\frac{1}{2}}}{(1-\eta^2)^{\frac{1}{2}}}\left\{\ddot{\eta} + \frac{2\xi}{(\eta^2+\xi^2)}\dot{\eta}\dot{\xi} + \frac{\eta(1+\xi^2)}{(1-\eta^2)(\eta^2+\xi^2)}\dot{\eta}^2 - \frac{\eta(1-\eta^2)}{(1+\xi^2)(\eta^2+\xi^2)}\dot{\xi}^2 + \frac{\eta(1-\eta^2)(1+\xi^2)}{(\eta^2+\xi^2)}\dot{\phi}^2\right\} \quad (23)$$

$$a_\xi = \frac{a(\eta^2+\xi^2)^{\frac{1}{2}}}{(1+\xi^2)^{\frac{1}{2}}}\left\{\ddot{\xi} - \frac{\xi(1+\xi^2)}{(1-\eta^2)(\eta^2+\xi^2)}\dot{\eta}^2 + \frac{\xi(1-\eta^2)}{(1+\xi^2)(\eta^2+\xi^2)}\dot{\xi}^2 + \frac{2\eta}{(\eta^2+\xi^2)}\dot{\eta}\dot{\xi} - \frac{\xi(1-\eta^2)(1+\xi^2)}{(\eta^2+\xi^2)}\dot{\phi}^2\right\} \quad (24)$$

$$\dot{a}_\phi = a(1-\eta^2)^{\frac{1}{2}}(1+\xi^2)^{\frac{1}{2}}\left\{\ddot{\phi} - \frac{2\eta}{(1-\eta^2)}\dot{\eta}\,\dot{\phi} + \frac{2\xi}{(1+\xi^2)}\dot{\zeta}\,\dot{\phi}\right\} \quad (25)$$

This is the completion of the theory of oblate spheroidal coordinate system.

SUMMARY

In this paper we derived the components of velocity and acceleration in oblate spheroidal coordinates as (19) – (21) and (22) – (24) respectively.

The results obtained in this paper are necessary and sufficient for expressing all mechanical quantities (linear momentum, kinetic energy, lagranian, Hamiltonian in terms of oblate spheroidal coordinates. Consequently, the way is cleared for expressing all dynamical laws of motion (Newton's law, Lagrange's law, Hamilton's law Einstein's special relativistic law of motion, and Schrodinger's law of quantum mechanics) entirely in terms of oblate spheroidal coordinates.

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