IN-DEPTH ANALYSES OF THE REWARD STRUCTURES OF N- PERSON COOPERATIVE GAMES WITH SPECIFIED WINNING COALITIONS INVOLVING AT LEAST ONE MAJOR PLAYER

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ABSTRACT: *Explicit computational rewards were obtained for n- person cooperative games involving specific classes of players of different ranks by the construction and deployment of cardinality-based listing structure of combined sets of winning coalitions. The results were rigorously analyzed and validated with clearly derived mathematical relationships.*

KEYWORDS: Cooperative, Coalition, Players, Rewards, Cardinality, Structure.

INTRODUCTION

Winston (1994) pointed out that Shapley value could be used as a measure of the power of individual members of a political or business organization; it was indicated in Winston (1994) that using a 0-1 characteristic function, it could be shown that 98.15% of the power in the Security Council resided with the permanent members. Kerby and Gobeler (1996) validated the assertion in Winston (1994) and much more using a functional set-theoretic cardinality approach that was rather involved. Ukwu (2014) obtained independent and much less esoteric proofs of the relevant results in Kerby and Gobeler (1996) by constructing and deploying a cardinality-based listing structure of combined winning coalitions from the sets of permanent and nonpermanent members; Ukwu (2014) pointed out that the later approach held a lot of promise for enhanced appreciation and extensions of the results to more general coalitional structures. This paper translates the extension potential of Ukwu (2014) to reality.

THEORETICAL UNDERPINNING

We first consider the reward structure of a general class of cooperative games among *n* players in which there is only one major player; this player must obtain the cooperation of some other players to achieve a set target, while all the other players put together cannot achieve the target without the cooperation of the major player. Next, we extend the result to two major players requiring certain levels of cooperation from other players to achieve an objective. Finally, we perform in-depth analyses of the reward structure associated with n- person cooperative games incorporating two major players of different ranks whose only coalition is non-winning.

The results have wide-ranging applications to resolution passing in meetings, distributions of political appointments, siting of projects and much more.

METHODOLOGY

Preliminaries

Let $N = \{1, 2, \cdots, n\}$

Definition

For each subset S of N, the characteristic function v gives the amount $v(S)$ those members of S can be sure of receiving if they act together and form a coalition.

Shapley Value Theorem

Given an *n*-person game with characteristic function *v*, there is a unique reward vector $x = (x_1, x_2, \cdots, x_n)$ satisfying axioms 1-4 stated below. The reward to the i^{th} player (x_i) is given by

$$x_i \;=\; \sum_{S\subset N:\, i\notin S} P_n\left[v(S\cup\{i\})-v(S)\right], \text{ where } p_n(S)=\frac{|S|!(n-1-|S|)!}{n!}$$

and $|S|$ is the number of players in the coalition *S*.
Axiom 1: Relabeling of players interchanges the players' rewards.

Axiom 2: $\sum_{i=1}^{n} x_i = v(N)$

Axiom 3: If $v(S-\{i\})=v(S)$ holds for all coalitions *S*, then the Shapley value has $x_i=0$. If player *i* adds no value to any coalition, player *i* receives a reward of zero from the Shapley value.
Axiom 4: Let *x* be the Shapley value vector for game *v* and *y* the Shapley value vector of game $\bar{v}$. Then the Shapley value vector for game $(v+\bar{v})$ is the vector *x* + *y*.
See Winston (1994) for the above theorem.

RESULTS

The first of the class of problems to be investigated can be couched in the following terms:

Theorem on one major-player cooperative game
Hypothesis**:** Consider an *n* - person cooperative game in which the winning coalitions are those coalitions involving one player and at least any j other players, where $1\le j\le n-1$, with a winning coalition receiving a reward of 1. How should the rewards be distributed among the players?
Conclusion**:** Under the standing hypothesis the major player should receive a numeric reward of $\frac{n-j}{n}$ or $1-\frac{j}{n}$, while the other players should each receive the reward $\frac{j}{n(n-1)}$.

Proof
Without loss of generality designate player 1 as the major player and let x_i be the reward to player $i, i\in\{1,2,\cdots,n\}$.Then we must prove that $x_1=\frac{n-j}{n}=1-\frac{j}{n}$ and $x_i=\frac{j}{n(n-1)}$, $i\in\{2,3,\cdots,n\}$.
Let *v* denote the characteristic function of the game and *S* an existing coalition.

Then $v(S)=\begin{cases}1, \text{ if } 1\in S \text{ and } j+1\le |S|\le n\\ 0, \text{ otherwise}\end{cases}$

Suppose that player *i* now joins the coalition. Then the remaining $n-1-|S|$ players arrive (join the coalition) after player *i*. Since the arrival process is random and there are *n* players involved, the probability of any order of arrival of the *n* players is $\frac{1}{n!}$. Denote this probability by P_n. Then the contribution of player i to the characteristic function of the coalition is $v(S\cup\{i\})-v(S)$. Set $N=\{1,2,\cdots,n\}$. Hence the reward to player i is given by

$$x_i \;=\; \sum_{i\notin S\subset N} P_n[v(S\cup\{i\}-v(S)]=\frac{1}{n!}\sum_{i\notin S\subset N}[v(S\cup\{i\}-v(S)]$$

In other words the reward to player i is the sum of values added to all prior coalitions for which player i is pivotal. Now, value is added by player 1 only if player 1 occupies position $t, j+1\le t\le n,$ since player 1 needs the cooperation of at least j other players. If player 1 occupies position *t*, then each of the remaining $n-1$ positions can be filled in exactly $(n-1)!$ ways. Hence for $1\notin S, v(S\cup\{1\})-v(S)=v(S\cup\{1\})=1$ in exactly $(n-1)!$ places in each of the t positions that player 1 may occupy, for $j+1\le t\le n.$ Hence

$$x_1=\frac{1}{n!}\sum_{t=j+1}^{n}(n-1)!=\frac{(n-1)!}{n!}(n-j)=\frac{n-j}{n}, \text{ proving the first part.}$$

Note that the value added by any other player in any of the above possible positions for player 1 is 0, since player 1 is a critical player. Now let $i\ne 1.$ For player i to add value, but player 1

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none, player i could not have arrived before player 1; moreover player i must appear in the $(j+1)^{st}$ position; needless to say that player 1 must have appeared in any of the first j positions before player i. Thereafter the remaining $n-2$ other positions must be filled in $(n-2)!$ ways.

Hence $x_i = \frac{1}{n!}\sum_{t=1}^{j}(n-2)! = \frac{(n-2)!}{n!}j = \frac{j}{n(n-1)}$, proving the second part.

Verification: $\sum_{i=1}^{n} x_i = \frac{n-j}{n} + (n-1)\frac{j}{n(n-1)} = 1$, as required.

Alternative proof of theorem 1

Value is added by player 1 only if it occupies position t, $j+1 \le t \le n$. That is, there must have been at least j other players prior to the arrival of player 1. Hence

$$x_1 = \sum_{|S|=j}^{n-1}\binom{n-1}{|S|}P_n(S) = \sum_{|S|=j}^{n-1}\binom{n-1}{|S|}|S|!\frac{(n-1-|S|)!}{n!} = \frac{1}{n}\sum_{t=j}^{n-1}\binom{n-1}{t}t!\frac{(n-1-t)!}{(n-1)!}$$

$$= \frac{1}{n}\sum_{t=j}^{n-1}\binom{n-1}{t}\binom{n-1}{t}^{-1} = \frac{n-j}{n},$$ proving the first part.

For $i \ne 1$ to add value, player i must arrive after player 1 and after the arrival of exactly j players. So, player i must be the $(j+1)^{st}$ player to arrive, leading to the assertion that $|S| = j$. With player i fixed in position $j+1$ and player 1 in any of the previous j positions, the remaining $j-1$ places for the other players may be selected in exactly $\binom{n-2}{j-1}$ ways.

Hence for $i \ne 1$, $x_i = \sum_{|S|=j}^{j}\binom{n-2}{j-1}P_n(S) = \frac{(n-2)!}{(n-1-j)!(j-1)!}\frac{j!(n-1-j)!}{n!}$

$$= \frac{(n-2)!\,j!}{n(n-1)(n-2)!} = \frac{j}{n(n-1)},$$ as stated.

Verification: We need to show that $\sum_{i=1}^{n} x_i = v(N)$. Clealy, $v(N) = 1$. Now, $\sum_{i=1}^{n} x_i = \frac{n-j}{n} + (n-1)\frac{j}{n(n-1)} = 1$, as desired.

Remarks: For $i \ne 1$, one could well have obtained x_i as follows:

$$x_i = \frac{1}{n-1}(1-x_1) = \frac{1}{n-1}\left[1-\frac{1}{n}(n-j)\right] = \frac{1}{n-1}\left[\frac{n-(n-j)}{n}\right] = \frac{j}{n(n-1)},$$ as desired.

However, this would not help in error - checking our results, as $\sum_{i=1}^{n} x_i$ would be equal to 1, automatically.

In the ensuing result, the above theorem will be extended to two major players requiring certain levels of cooperation from other players to achieve an objective. The statement of the problem and its solutions are encapsulated in the following theorem.

Theorem on *n* - person cooperative game with two major players

Consider an n - person cooperative game in which the winning coalitions are those coalitions involving the following:

- Player 1 and player 2
- Player 1 with at least j other players, excluding player 2, where $1 \le j \le n-2$, in a coalition of cardinality at most n - 1
- Player 2 with all the other $n-2$ players excluding player 1

Let $N_k = \{k, k+1, \cdots, n\}$. For $i \in N_1$, let x_i be the reward to player i in all alliances involving player i. Then

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$$x_1 = \frac{1}{2n(n-1)}[2n^2 - 2nj - 4n + j^2 + j + 2]$$

$$x_2 = \frac{1}{2n(n-1)}[j^2 + j + 2]$$

$$x_i = \frac{1}{2n(n-1)(n-2)}[2n + 2nj - 2j^2 - 2j - 4], \text{ for } i \in N_3.$$

Observe that $x_1 = x_2 + \frac{n-j-2}{n-1} > x_2$. Also, $i \in N_3 \Rightarrow x_i = \frac{1}{n-2}\left[-2x_2 + \frac{1+j}{n-1}\right]$

Proof

Let v be the characteristic function of the game and let S be a coalition of players. Then

$v(\{i\}) = 0,\ \forall i \in N_3,\ v(\{1,2\}) = 1,\ v(N_1) = 1$

$$v(S) = \begin{cases} 0, \text{ if } |S| \le j, \text{ or } \{1,2\} \not\subset S \text{ or } |S| \le n-2 \text{ and } 1 \notin S \\ 1, \text{ if } j+1 \le |S| \le n \text{ and } 1 \in S \text{ or } |S| = n-1 \text{ and } 2 \in S \end{cases}$$

We proceed by first computing the Shapley values to players 1 and 2. Let C_t denote a coalition of t players, excluding players 1 and 2; let $\tilde{C}_t$ be a coalition of size t from the set N_3, excluding one player from the set and let $\tilde{S}_t$ be a relevant set of cardinality t.

The key to achieving the proof lies in laying out an appropriate tabular structure involving all coalitions S for which $v(S \cup \{i\}) - v(S) = 1$, the probability, $P_n(S)$ of occurrence of S prior to the arrival of player i and the number of ways of selecting S, denoted by $N(S)$

A tabular structure for computing player 1's reward.

S:	C_j	C_{j+1}	$\ldots C_{n-2}$	$\{2\}$	$C_1 \cup \{2\}$	$\ldots C_{n-3} \cup \{2\}$
$P_n(S)$:						
$N(S)$						

where $P_n(S) = \frac{|S|!(n-|S|-1)!}{n!}$; $N(S) = \begin{cases} \binom{n-2}{|S|}, 2 \notin S \\ \binom{n-2}{|S|-1}, 2 \in S \end{cases}$

Therefore,

$$x_1 = \sum_{|S|=j}^{n-2} \binom{n-2}{|S|} P_n(S) + \sum_{t=1}^{n-3} \binom{n-2}{t} P_n(\tilde{S}_{t+1}) + P_n(\{2\})$$

$$= \sum_{t=j}^{n-2} \binom{n-2}{t} \frac{t!(n-t-1)!}{n!} + \sum_{t=1}^{n-3} \binom{n-2}{t} \frac{(t+1)t!(n-t-2)!}{n!} + \frac{(n-2)!}{n!}$$

$$= \sum_{t=j}^{n-2} \frac{n-t-1}{n(n-1)} + \sum_{t=1}^{n-3} \frac{t+1}{n(n-1)} + \frac{1}{n(n-1)} = \sum_{t=1}^{n-2} \frac{n-t-1}{n(n-1)} - \sum_{t=1}^{j-1} \frac{n-t-1}{n(n-1)} + \sum_{t=1}^{n-3} \frac{t+1}{n(n-1)} + \frac{1}{n(n-1)}$$

$$= \frac{1}{n(n-1)}\left[(n-1)(n-2) - \frac{(n-1)(n-2)}{2} - (n-1)(j-1) + \frac{j(j-1)}{2}\right]$$

$$= \frac{1}{2n(n-1)}\left[2n^2 - 2nj - 4n + j^2 + j + 2\right], \text{ as stated.}$$

Player 2's reward: The appropriate listing for S is as displayed below.

S: C_{n-2}, $\{1\}$, $C_1 \cup \{1\}$, $\ldots, C_{j-1} \cup \{1\}$

Note that for $j \le t \le n-2$, $C_t \cup \{1\}$ is a winning coalition, adding value only to player 1; so it had to be excluded in the listing for player 2's reward. Hence

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$$x_2 = \frac{(n-2)!}{n!} + \sum_{t=1}^{j-1} \binom{n-2}{t} P_n(\tilde{S}_{t+1}) + P_n(C_{n-2})$$

$$= \frac{2(n-2)!}{n!} + \sum_{t=1}^{j-1} \frac{(n-2)!}{(n-t-2)!\,t!} \frac{(t+1)\,t!\,(n-t-2)!}{n!} = \frac{2}{n(n-1)} + \sum_{t=1}^{j-1} \frac{t+1}{n(n-1)}$$

$$= \frac{1}{n(n-1)}\left[2 + \sum_{t=1}^{j-1}(t+1)\right] = \frac{1}{n(n-1)}\left[2 + \frac{j(j-1)}{2} + j - 1\right] = \frac{1}{2n(n-1)}\left[j^2 + j + 2\right],$$ as postulated.

For $i \in N_3$, the relevant S sets for direct computation of x_i are $S:\ \tilde{C}_{j-1} \cup \{1\}$ and $\tilde{C}_{n-3} \cup \{2\}$, noting that for $j \le t \le n-2$, $\tilde{C}_t \cup \{1\}$ is a winning coalition adding value only to player 1. Consequently

$$x_i = \sum_{|S|=j}^{j} \binom{n-3}{|S|-1} P_n(S) + \sum_{|S|=n-2}^{n-2} \binom{n-3}{|S|-1} P_n(C_{n-3} \cup \{2\})$$

$$= \binom{n-3}{j-1} \frac{j!(n-j-1)!}{n!} + \frac{(n-2)!}{n!} = \frac{j(n-j-1)}{n(n-1)(n-2)} + \frac{1}{n(n-1)} = \frac{1}{2n(n-1)}\left[2 + \frac{2j(n-j-1)}{n-2}\right]$$

$$= \frac{1}{2n(n-1)(n-2)}\left[\,2n + 2j - 2j^2 - 2j - 4\right],$$ as required.

DISCUSSION

For practical purposes one needs to establish that all rewards are non-negative and sum to 1 by appropriate probability axioms.

Feasibility and verification of results

We need to show that the rewards are all nonnegative and sum to 1.

Sum to 1 requirement

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + (n-2)x_t;\ t \in N_3$$

$$\Rightarrow \sum_{i=1}^{n} x_i = x_1 + x_2 + (n-2)\frac{1}{2n(n-1)(n-2)}\left[2n - 4 + 2nj - 2j^2\right]$$

$$= \frac{1}{2n(n-1)}\left[2n^2 - 2nj - 4n + j^2 + j + 2 + j^2 + j + 2 + 2n - 4 + 2nj - 2j^2 - 2j\right]$$

$$= \frac{1}{2n(n-1)}\left[2n^2 - 2n\right] = 1,$$ as desired.

Non-negativity requirements on voting powers

$$x_1 = \frac{1}{2n(n-1)}[2n^2 - 2nj - 4n + j^2 + j + 2].$$

$2n^2 - 2nj - 4n + j^2 + j + 2 = 2n(n-j-2) + j^2 + j + 2 > 4,$ since $1 \le j \le n-2$.

Therefore $x_1 > 0$. Clearly $x_2 > 0$. Now, $i \in N_3 = \{3, 4, \ldots, n\} \Rightarrow$

$$x_i = \frac{1}{2n(n-1)(n-2)}[2n + 2nj - 2j^2 - 2j - 4].$$

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$2n+2nj-2j^2-2j-4=2\left[j(n-j)+\left((n-j-2)\right)\right]>4$. Therefore, $x_i>0$. This implies that there are no dummies in the coalitional structure; needless to say: there are no dictators or veto players.

Observations/Analysis

$x_1=x_2+\dfrac{n-j-2}{n-1}\geq x_2;\, i\in N_3 \Rightarrow x_i=-\dfrac{2}{n-2}x_2+\dfrac{j+1}{(n-1)(n-2)}$.

$x_2\geq x_i$, with equality iff $j=1$. To see this note that $x_2\geq x_i$, iff $(n-2)\left[j^2+j+2\right]\geq 2n-4+2nj-2j^2-2j$. This is true iff $nj^2+nj\geq 2nj$.

Clearly, $nj^2+nj=nj(j+1)\geq 2nj$, with equality iff $j=1$.

The reader can verify that at $j=1$, x_2 and x_i coincide with the common value $\dfrac{2}{n(n-1)}$. This is consistent with the hypothesis of the theorem which for $j=1$, recognizes only player 1 as a major player; other players are given the same lower status. This case also coincides with theorem 1 with j replaced by $j+1$ there, in line with the structure of theorem 2 which can only be compared with theorem 1 when $j=1$.

Also x_1 and x_2 coincide only for $j=n-2$. This should be expected since the hypothesis of theorem would accord the same status to players 1 and 2 when $j=n-2$.

Conclusion: $i\in\{3,4,\ldots n\}\Rightarrow x_1>x_2>x_i;\ j\notin\{1,n-2\}$.

We will now extend above result to a situation where player 2 could benefit only in coalition with player 1 and k other players, where $k< j-1$. The problem is summarized in the ensuing theorem with accompanying proof.

Theorem on an *n* - person cooperative games with power relaxation to two major players

Consider an n – person cooperative game in which the winning coalitions are those coalitions involving the following:

- Player 1 with at least j other players, excluding player 2, where $1\leq j\leq n-2$
- Player 2 with player 1 and at least k other players, where $1\leq k\leq j-2$
- Player 2 with all the other $n-2$ players excluding player 1

Let $N_k=\{k,k+1,\cdots,n\}$. For $i\in N_1$, let x_i be the reward to player *i* in all alliances involving player *i*. Then

$$x_1=\frac{1}{2n(n-1)}[2n^2-2nj-4n+j^2+j+2-k^2-k]$$

$$x_2=\frac{1}{2n(n-1)}[j^2+j+2-k^2-k]=\frac{1}{2n(n-1)}[(j-k)(j+k+1)+2]$$

$$x_i=\frac{1}{2n(n-1)(n-2)}[2n+2nj-2j^2-2j-4+2k^2+2k],\text{ for } i\in N_3.$$

Observe that $x_1=x_2+\dfrac{n-j-2}{n-1}\geq x_2$. Also, $i\in N_3\Rightarrow x_i=\dfrac{1}{n-2}\left[-2x_2+\dfrac{1+j}{n-1}\right]$

Therefore, the rewards maintain the same relationship structure as in theorem 2.

Proof

We begin with the computation of player 1's reward.

The relevant S values are $C_j, C_{j+1},\ldots,C_{n-2}, C_k\cup\{2\}, C_{k+1}\cup\{2\},\ldots,C_{n-3}\cup\{2\}$.

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Hence,

$$x_1 = \sum_{|S|=j}^{n-2} \binom{n-2}{|S|} P_n(S) + \sum_{t=k}^{n-3} \binom{n-2}{t} P_n(\tilde{S}_{t+1})$$

$$= \sum_{t=j}^{n-2} \binom{n-2}{t} \frac{t!(n-t-1)!}{n!} + \sum_{t=k}^{n-3} \binom{n-2}{t} \frac{(t+1)!(n-t-2)!}{n!}$$

$$= \sum_{t=j}^{n-2} \frac{(n-t-1)}{n(n-1)} + \sum_{t=k}^{n-3} \frac{(t+1)}{n!} = \frac{1}{n(n-1)}\left[\sum_{t=1}^{n-2}(n-t-1) - \sum_{t=1}^{j-1}(n-t-1) + \sum_{t=1}^{n-3}(t+1) - \sum_{t=1}^{k-1}(t+1)\right]$$

$$= \frac{1}{n(n-1)}\left[\begin{array}{l}(n-1)(n-2) - \frac{(n-1)(n-2)}{2} - (n-1)(j-1) + \frac{j(j-1)}{2} \\ + n - 3 + \frac{(n-3)(n-2)}{2} - (k-1) - \frac{(k-1)k}{2}\end{array}\right]$$

$$\frac{1}{2n(n-1)}\left[2n^2 + j^2 - k^2 - 2nj - 4n + j - k + 2\right], \text{ as stated.}$$

For player 2's reward, the relevant S sets are $C_k \cup \{1\}, C_{k+1} \cup \{1\}, \ldots, C_{j-1} \cup \{1\}$, and C_{n-2}, noting that $C_t \cup \{1\}$ is a winning coalition for $t \geq j$. Therefore, we must restrict the set of coalitions involving player 1 to $C_t \cup \{1\}$, $k \leq t \leq j-1$in order to add value to player 2. Hence

$$x_2 = \sum_{t=k}^{j-1} \binom{n-2}{t} P_n(\tilde{S}_{t+1}) + \frac{(n-2)!}{n!} = \sum_{t=k}^{j-1} \binom{n-2}{t} \frac{1}{t+1}\left[{}^{n}C_{t+1}\right]^{-1}$$

$$= \sum_{t=k}^{j-1} \frac{(n-2)!}{(n-t-2)!\,t!} \frac{(t+1)!(n-t-2)!}{n!} + \frac{1}{n(n-1)} = \frac{1}{n(n-1)}\left[1 + \sum_{t=k}^{j-1}(t+1)\right]$$

$$\frac{1}{n(n-1)}\left[1 + j - k + \frac{(j-1)j}{2} - \frac{(k-1)k}{2}\right] = \frac{1}{2n(n-1)}\left[j^2 - k^2 + j - k + 2\right], \text{as desired.}$$

For $i \in \{3, 4, \ldots, n\}$, the appropriate S sets for player i's reward are $\tilde{C}_{j-1} \cup \{1\}$, $\tilde{C}_{n-3} \cup \{2\}$, and $\tilde{C}_{k-1} \{1, 2\}$. Hence

$$x_i = \binom{n-3}{j-1} \frac{j!(n-j-1)!}{n!} + \frac{(n-2)!}{n!} + (n-3)! \frac{(k+1)!(n-k-2)!}{n!}$$

$$= \frac{j(n-j-1)}{n(n-1)(n-2)} + \frac{1}{n(n-1)} + \frac{(k+1)k}{n(n-1)(n-2)}$$

$$= \frac{1}{2n(n-1)(n-2)}\left[2nj - 2j^2 - 2j + 2n - 4 + 2k^2 + 2k\right].$$

Observations/Analysis

Observe that $x_1 = x_2 + \dfrac{n-j-2}{n-1} \geq x_2$. Also, $i \in N_3 \Rightarrow x_i = \dfrac{1}{n-2}\left[-2x_2 + \dfrac{1+j}{n-1}\right]$

This preserves the reward structure of theorem 2. Also note the following relationships among corresponding rewards in theorem 2 and 3:

$$(x_1 \text{ in theorem 3}) = (x_1 \text{ in theorem 2}) - \frac{k(k+1)}{2n(n-1)}$$

$$(x_2 \text{ in theorem 3}) = (x_2 \text{ in theorem 2}) - \frac{k(k+1)}{2n(n-1)}$$

x_1 and x_2 lost the same value $\dfrac{k(k+1)}{2n(n-1)}$. The fact that they lost some value is hardly surprising, since the powers they had in theorem 2 were diluted in theorem 3. What is quite

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striking /exciting is the fact that they lost exactly the same value of magnitude $\frac{k(k+1)}{2n(n-1)}$. If there's any justice in the world then for players $i \in \{3, 4, \ldots, n\}$, who acquired more powers in theorem 3, these x_i 's put together should gain the total value lost by players 1 and 2. In deed,

$$(x_i \text{ in theorem } 3) = (x_i \text{ in theorem } 2) + \frac{2k(k+1)}{2n(n-1)(n-2)}$$

$$\text{Total gain by the } x_i\text{'s} = (n-2)\frac{2k(k+1)}{2n(n-1)(n-2)} = \frac{k(k+1)}{n(n-1)} = \text{total loss by } x_1 \text{ and } x_2.$$

This was exactly what we wished for.

Illustrative examples

(i) Suppose that decisions are made by majority rule in a body consisting of A, B, C, and D, who have 3, 2, 1, and 1 votes, respectively. The majority vote threshold is 4. Determine the rewards or voting powers of the players.

Solution

Theorem 4.2 is relevant, with A identified as player 1, B player 2; players C and D as players 3 and 4. Therefore $n = 4, j = 1 \Rightarrow x_1 = \frac{1}{8(3)}[32-8-16+1+1+2] = \frac{1}{2},\ x_2 = \frac{1}{8(3)}[1+1+1] = \frac{1}{6}$,

$$x_3 = x_4 = \frac{1}{8(3)(2)}[8+8-2-2-4] = \frac{1}{6}$$

Therefore A has 50 percent voting power; B, C and D each has a voting power of one-sixth or about 16.67 percent.

(ii) On a college's basketball team, the decision of whether a student is allowed to play is made by four people: the head coach and the three assistants. To be allowed to play, the student needs approval from the head coach and at least one assistant coach. Determine the rewards or decision-making powers of the head coach and the assistants.

Solution

Apply theorem 4.1, with $n = 4$, together with the following identifications: the head coach $\leftrightarrow$ player 1, the assistants $\leftrightarrow$ players 2,3 and 4; $j = 1$. Therefore $x_1 = \frac{n-j}{n} = \frac{4-1}{4} = \frac{3}{4},\ x_2 = x_3 = x_4 = \frac{1}{4(3)} = \frac{1}{12}$.

Therefore the head coach has 75 percent of the powers, leaving each of the assistants with about 8.33 percent decision-making powers.

(iii) An executive board consists of a president (P) and three vice-presidents $(\mathrm{V}_1, \mathrm{V}_2, \mathrm{V}_3)$. For a motion to pass it must have three yes votes, one of which must be the president's. Determine the voting rights of the four stake-holders.

Solution

Theorem 4.1 is relevant, with the president identified as player 1 and the three vice-presidents as players 2, 3 and 4. players 2, 3 and 4. Therefore $n = 4, j = 2 \Rightarrow x_1 = \frac{n-j}{n} = \frac{4-2}{4} = \frac{1}{2},\ x_2 = x_3 = x_4 = \frac{1}{6}$.

Therefore the president has 50 percent voting power; each of the three vice-presidents has a voting power of one-sixth or about 16.67 percent.

(iv) In a corporation, the shareholders receive 1 vote for each share of stock they hold, which is usually based on the amount of money they invested in the company. Suppose a small corporation has six people who invested \$40,000, \$35,000, \$25,000, \$20,000, \$15,000, \$15,000. If they receive one share of stock for each \$1000 invested, and any decisions require two-third votes cast, set up a weighted voting system to represent the corporation's shareholder votes and hence determine the voting powers of the shareholders.

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Solution

The voting system has the representation $\left[q; w_1, w_2, \cdots w_n\right] = [100; 40, 35, 25, 20, 15, 15]$, noting $n = 6$ and that a two-third majority threshold vote or quota, q is 100. Apply theorem 5.3, with $n = 6,\ j = 3$:

$j = 3 \Rightarrow k = 1 \Rightarrow$

$$x_1 = \frac{1}{2n(n-1)}[2n^2 - 2n\,j - 4n + j^2 + j + 2 - k^2 - k] = \frac{1}{60}[72 - 36 - 24 + 9 + 3 + 2 - 1 - 1] = \frac{2}{5}$$

$$x_2 = \frac{1}{2n(n-1)}[j^2 + j + 2 - k^2 - k] = \frac{1}{60}[(j-k)(j+k+1) + 2] = \frac{1}{60}\left[2(5) + 2\right] = \frac{1}{5}$$

$$x_i = \frac{1}{2n(n-1)(n-2)}[2n + 2n\,j - 2j^2 - 2j - 4 + 2k^2 + 2k] = \frac{1}{240}[12 + 36 - 18 - 6 - 4 + 2 + 2] = \frac{1}{10},$$

for $i \in N_3 = \{3, 4, 5, 6\}$.

Therefore the highest investor has 40 percent voting power, followed by the next highest investor, with 20 percent voting power; each of the other four investors has a voting power of 10 percent.

IMPLICATIONS TO RESEARCH AND PRACTICE

The established results have wide-ranging implications and applications to resolution passing in meetings, distributions of political appointments, executive and parliamentary decisions, siting of projects, contract negotiations, and sporting team compositions, to mention just a few.

CONCLUSION

This paper deployed cardinality dependent sequential coalitional structures to perform in-depth analyses of the reward structures of three n - person cooperative games with specified winning coalitions involving at least one major player, for various quota configurations, based on the Shapley Value reward concept.The reward expressions in all three problems are found to be consistent, with clearly determined relationships and structures and satisfy implicit, probabilistic and practical feasibility conditions. The results may be appropriated to give prescriptions for equitable allocations and distribution of political appointments, siting of projects and voting rights for resolution passing in public or corporate meetings. In a follow- up paper the method developed here will be exploited and extended to derive reward structures for players in more general cooperative game settings.

FUTURE RESEARCH

Further research interest and extensions of this article will include the following investigations:

(i) Reward structure of finite - person cooperative games with a lone player category and two other categories of players subject to specified winning coalitions.

(ii) Reward structure of finite - person cooperative games with three broad categories of players subject to specified winning coalitions.

(iii) Analyses of three - category, finite - person cooperative games with relaxation in winning coalitions.

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