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The structure of determining matrices for a class of double – delay control systems

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ABSTRACT: *This paper derived and established the structure of determining matrices for a class of double – delay autonomous linear differential systems through a sequence of lemmas, theorems, corollaries and the exploitation of key facts about permutations. The proofs were achieved using ingenious combinations of summation notations, the multinomial distribution, the greatest integer function, change of variables technique and compositions of signum and max functions.*
The paper has extended the results on single–delay models, with more complexity in the structure of the determining matrices.

KEYWORDS: *Delay, Determining, Double, Structure, Systems.*

I. INTRODUCTION

The importance of determining matrices stems from the fact that they constitute the optimal instrumentality for the determination of Euclidean controllability and compactness of cores of Euclidean targets. See Gabasov and Kirillova (1976) and Ukwu (1992, 1996, 2013a). In sharp contrast to determining matrices, the use of indices of control systems on the one hand and the application of controllability Grammians on the other, for the investigation of the Euclidean controllability of systems can at the very best be quite computationally challenging and at the worst, mathematically intractable. Thus, determining matrices are beautiful brides for the interrogation of the controllability disposition of delay control systems. Also see Ukwu (2013a).

However up-to-date review of literature on this subject reveals that there is currently no result on the structure of determining matrices for double-delay systems. This could be attributed to the severe difficulty in identifying recognizable mathematical patterns needed for inductive proof of any claimed result. Thus, this paper makes a positive contribution to knowledge by correctly establishing the structure of such determining matrices in this area of acute research need.

II. MATERIALS AND METHODS

The derivation of necessary and sufficient condition for the Euclidean controllability of system (1) on the interval $[0,\ t_1]$, using determining matrices depends on

1) obtaining workable expressions for the determining equations of the $n\times n$ matrices $Q_k(jh)$, for $j: t_1 - jh > 0, k = 0, 1, \cdots$
2) showing that $\Delta X^{(k)}(t_1 - jh, t_1) = (-1)^k Q_k(jh)$, for j: $t_1 - jh > 0, k = 0, 1, \ldots$
3) where $\Delta X^{(k)}(t_1 - jh, t_1) = X^{(k)}((t_1 - jh)^-, t_1) - X^{(k)}((t_1 - jh)^+, t_1)$
4) showing that $Q_\infty(t_1)$ (t_1) is a linear combination of $Q_0(s), Q_1(s), \cdots, Q_{n-1}(s); s = 0, h, \cdots (n-1)h.$

See Ukwu (2013a).

Our objective is to prosecute task (i) in all ramifications. Tasks (ii) and (iii) will be prosecuted in other papers.

2.1 Identification of Work-Based Double-Delay Autonomous Control System

We consider the double-delay autonomous control system:

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + A_2 x(t-2h) + B\,u(t);\ t \geq 0 \qquad (1)$$

$$x(t) = \phi(t),\ t \in [-2h, 0],\ h > 0 \qquad (2)$$

Where A_0, A_1, A_2 are $n \times n$ constant matrices with real entries, B is an $n \times m$ constant matrix with real entries. The initial function ϕ is in $C\left(\left[-2h, 0\right], \mathbf{R}^n\right)$, the space of continuous functions from $[-2h, 0]$ into the real n-dimension Euclidean space, $\mathbf{R}^n$ with norm defined by $\|\phi\| = \sup_{t \in [-2h, 0]} |\phi(t)|$, (the sup norm). The control u is in the space $L_\infty\left(\left[0, t_1\right], \mathbf{R}^n\right)$, the space of essentially bounded measurable functions taking $\left[0, t_1\right]$ into $\mathbf{R}^n$ with norm $\|\phi\| = \operatorname{ess\,sup}_{t \in [0, t_1]} |u(t)|$.

Any control $u \in L_\infty\left(\left[0, t_1\right], \mathbf{R}^n\right)$ will be referred to as an admissible control. For full discussion on the spaces C^{p-1} and L_p (or L^p), $p \in \{1, 2, ..., \infty\}$, see Chidume (2003 and 2007) and Royden (1988).

2.2 Preliminaries on the Partial Derivatives $\dfrac{\partial^k X(\tau, t)}{\partial \tau^k}, k = 0, 1, \cdots$

Let $t, \tau \in \left[0, t_1\right]$. For fixed t, let $\tau \to X(\tau, t)$ satisfy the matrix differential equation:

$$\frac{\partial}{\partial \tau} X(\tau, t) = -X(\tau, t) A_0 - X(\tau + h, t) A_1 - X(\tau + 2h, t) A_2 \tag{3}$$

for $0 < \tau < t, \tau \neq t - kh, k = 0, 1, ...$ where $X(\tau, t) = \begin{cases} I_n; \tau = t \\ 0; \tau > t \end{cases}$

See Chukwu (1992), Hale (1977) and Tadmore (1984) for properties of $X(t, \tau)$. Of particular importance is the fact that $\tau \to X(\tau, t)$ is analytic on the intervals $\left(t_1 - (j+1)h, t_1 - jh\right), j = 0, 1, ..., t_1 - (j+1)h > 0$. Any such $\tau \in \left(t_1 - (j+1)h, t_1 - jh\right)$ is called a regular point of $\tau \to X(t, \tau)$. See also Analytic function (2010) for a discussion on analytic functions. Let $X^{(k)}(\tau, t)$ denote $\dfrac{\partial^k}{\partial \tau^k} X(\tau, t_1)$, the k^{th} partial derivative of $X(\tau, t_1)$ with respect to τ, where τ is in $\left(t_1 - (j+1)h, \ t_1 - jh\right); j = 0, 1, ..., r$, for some integer r such that $t_1 - (r+1)h > 0$. Write $X^{(k+1)}(\tau, t_1) = \dfrac{\delta}{\delta \tau} X^k(\tau, t_1)$.

Define:

$$\Delta X^{(k)}(t_1 - jh, t_1) = X^{(k)}\left(t_1, (t_1 - jh)^-, t_1\right) - X^{(k)}\left((t_1 - jh)^+, t_1\right), \tag{4}$$

for $k = 0, 1, ...; \ j = 0, 1, ...; t_1 - jh > 0,$

where $X^{(k)}\left((t_1 - jh)^-, t_1\right)$ and $X^{(k)}\left(t_1, (t_1 - jh)^+, t_1\right)$ denote respectively the left and right hand limits of $X^{(k)}(\tau, t_1)$ at $\tau = t_1 - jh$. Hence:

$$X^{(k)}\left((t_1 - jh)^-, t_1\right) = \lim_{\substack{\tau \to t_1 - jh \\ t_1 - (j+1)h < \tau < t_1 - jh}} X^{(k)}(\tau, t_1) \tag{5}$$

$$X^{(k)}\left((t_1 - jh)^+, t_1\right) = \lim_{\substack{\tau \to t_1 - jh \\ t_1 - jh < \tau < t_1 - (j-1)h}} X^{(k)}\left(\tau, t_1\right) \tag{6}$$

2.3 Definition, Existence and Uniqueness of Determining Matrices for System (1)

Let Q_k (s) be then $n \times n$ matrix function defined by:

$$Q_k(s) = A_0 Q_{k-1}(s) + A_1 Q_{k-1}(s-h) + A_2 Q_{k-1}(s-2h) \tag{7}$$

for $k = 1, 2, \cdots;\ s > 0,$ with initial conditions:

$$Q_0(0) = I_n \tag{8}$$

$$Q_0(s) = 0;\ s \neq 0 \tag{9}$$

These initial conditions guarantee the unique solvability of (7). Cf. [1].

The stage is now set for the establishment of the expressions and the structure of the determining matrices for system (1), as well as their relationships with $X^{(k)}(t,\tau)$ through a sequence of lemmas, theorems and corollaries and the exploitation of key facts about permutations.

2.4 Lemma on permutation products and sums

Let r_0, r_1, r_2 be nonnegative integers and let $P_{0(r_0),1(r_1),2(r_2)}$ denote the set of all permutations of $\underbrace{0,0,...0}_{r_0 \text{ times}}\ \underbrace{1,1,...1}_{r_1 \text{ times}}\ \underbrace{2,2,...2}_{r_2 \text{ times}}$: the permutations of the objects $0, 1,$ and 2 in which i appears r_i times, $i \in \{0,1,2\}$.

Let $P^{iL}_{0(r_0),1(r_1),2(r_2)}$ denote the subset of $P_{0(r_0),1(r_1),2(r_2)}$ with leading i, that is, those with i occupying the first position. Let $P^{iT}_{0(r_0),1(r_1),2(r_2)}$ denote the subset of $P_{0(r_0),1(r_1),2(r_2)}$ with trailing i, that is, those with i occupying the last position. Set $r = r_0 + r_1 + r_2$. Then for any fixed $r_0,\ r_1,\ r_2$,

(a) $$\sum_{(v_1,...v_r) \in P_{0(r_0),1(r_1),1(r_2)}} A_{v_1} ..., A_{v_r} = S_r(r_0, r_1, r_2) = \sum_{i=0}^{2} \sum_{(v_1,...v_r) \in P^{iL}_{0(r_0),1(r_1),1(r_2)}} A_{v_1} ..., A_{v_r} = \sum_{i=0}^{2} S_r^{iL}(r_0, r_1, r_2)$$

$$= A_0 \left[\sum_{(v_1,...v_{r-1}) \in P_{0(\max\{0, r_0-1\}),1(r_1),1(r_2)}} A_{v_1} ..., A_{v_{r-1}} \right] \operatorname{sgn}(r_0) + A_1 \left[\sum_{(v_1,...v_{r-1}) \in P_{0(r_0),1(\max\{0, r_1-1\}),1(r_2)}} A_{v_1} ..., A_{v_{r-1}} \right] \operatorname{sgn}(r_1)$$

$$+ A_2 \left[\sum_{(v_1,...v_{r-1}) \in P_{0(r_0),1(r_1),2(\max\{0, r_2-1\})}} A_{v_1} ..., A_{v_{r-1}} \right] \operatorname{sgn}(r_2)$$

(b) $$\sum_{(v_1,...v_r) \in P_{0(r_0),1(r_1),1(r_2)}} A_{v_1} ..., A_{v_r} = S_r(r_0, r_1, r_2) = \sum_{i=0}^{2} \sum_{(v_1,...v_r) \in P^{iT}_{0(r_0),1(r_1),1(r_2)}} A_{v_1} ..., A_{v_r} = \sum_{i=0}^{2} S_r^{iT}(r_0, r_1, r_2)$$

$$= \left[\sum_{(v_1,...v_{r-1}) \in P_{0(\max\{0, r_0-1\}),1(r_1),1(r_2)}} A_{v_1} ..., A_{v_{r-1}} \right] A_0 \operatorname{sgn}(r_0) + \left[\sum_{(v_1,...v_{r-1}) \in P_{0(r_0),1(\max\{0, r_1-1\}),1(r_2)}} A_{v_1} ..., A_{v_{r-1}} \right] A_1 \operatorname{sgn}(r_1)$$

$$+ \left[\sum_{(v_1,...v_{r-1}) \in P_{0(r_0),1(r_1),2(\max\{0, r_2-1\})}} A_{v_1} ..., A_{v_{r-1}} \right] A_2 \operatorname{sgn}(r_2)$$

(c) Hence for all nonnegative integers r_0, r_1, r_2 such that $r = r_0 + r_1 + r_2$,

$$\sum_{\substack{(v_1,\ldots v_r)\in P_{0(r_0),1(r_1),1(r_2)}\\ r_0+r_1+r_2=r}} A_{v_1}\ldots, A_{v_r} = S_r(r_0,r_1,r_2) = \sum_{i=0}^{2} \sum_{\substack{(v_1,\ldots v_r)\in P^{iL}_{0(r_0),1(r_1),1(r_2)}\\ r_0+r_1+r_2=r}} A_{v_1}\ldots, A_{v_r}$$

$$= \sum_{i=0}^{2} \sum_{r_0+r_1+r_2=r} S_r^{iL}(r_0,r_1,r_2)$$

Similar statements hold with respect to the remaining relations. Note that

$$\sum_{i=0}^{2} \sum_{\substack{(v_1,\ldots v_r)\in P^{iL}_{0(r_0),1(r_1),2(r_2)}\\ r_0+r_1+r_2=r}} A_{v_1}\ldots, A_{v_r} = \sum_{i=0}^{2}\sum_{r_0=0}^{r}\sum_{r_1=0}^{r-r_0} \sum_{(v_1,\ldots v_r)\in P^{iL}_{0(r_0),1(r_1),2(r_2)}} A_{v_1}\ldots, A_{v_r}$$

$\mathrm{sgn}(r_i)$ ensures that the corresponding expression vanishes if $r_i = 0 \Rightarrow A_i$ does not appear and so cannot be factored out. $\max\{0, r_i - 1\}$ ensures that the resulting permutations are well-defined.

In order not to clutter the work with '$\max\{0, r_i - 1\}$' and '$\mathrm{sgn}(r_i)$', the standard convention of letting

$$\sum_{(v_1,\ldots v_r)\in P_{0(\tilde r_0),1(\tilde r_1),1(\tilde r_2)}} A_{v_1}\ldots, A_{v_r} = 0, \text{ for any fixed } \tilde r_0, \tilde r_1, \tilde r_2;\ \tilde r_0+\tilde r_1+\tilde r_2 = r : \tilde r_i < 0, \text{for some } i\in\{0,1,2\}$$

would be adopted, as needed.

Proofs of (a), (b) and (c)

Every permutation involving 0, 1, and 2 must be led by one of those objects. If 0, 1 and 2 appear at least once, then each of them must lead at least once. Equivalent statements hold with 'led' replaced by 'trailed' and 'lead' replaced by 'trail'. Hence the sum of the products of the permutations must be the sum of the products of those permutations led (trailed) by A_0, A_1, and A_2 respectively. Consequently,

$$\sum_{r_0=0}^{r}\sum_{r_1=0}^{r-r_0} \sum_{\substack{(v_1,\ldots v_r)\in P_{0(r_0),1(r_1),2(r-r_0-r_1)}\\ r_0+r_1+r_2=r}} A_{v_1}\ldots, A_{v_r} = \sum_{r_0=0}^{r}\sum_{r_1=0}^{r-r_0} S_r(r_0,r_1,r-r_0-r_1)$$

$$=\sum_{i=0}^{2}\left(\sum_{r_0=0}^{r}\sum_{r_1=0}^{r-r_0} S_r(r_0,r_1,r-r_0-r_1)\right) \text{restricted to those permutations with leading(trailing } A_i)$$

$$= A_0S_{r-1}(\max\{0,r_0-1\},r_1,r_2)\mathrm{sgn}(r_0) + A_1S_{r-1}(r_0,\max\{0,r_1-1\},r_2)\mathrm{sgn}(r_1)$$
$$+ A_2S_{r-1}(r_0,r_1,\max\{0,r_2-1\})\mathrm{sgn}(r_2)$$
$$= S_{r-1}(\max\{0,r_0-1\},r_1,r_2)A_0\,\mathrm{sgn}(r_0) + S_{r-1}(r_0,\max\{0,r_1-1\},r_2)A_1\,\mathrm{sgn}(r_1)$$
$$+ S_{r-1}(r_0,r_1,\max\{0,r_2-1\})A_2\,\mathrm{sgn}(r_2)$$

2.5 Preliminary Lemma on Determining Matrices $Q_k(s), s\in\mathbf{R}$

(i) $Q_k(0) = A_0^k$

(ii) $Q_k(s) = 0$ if $s \neq rh$ for any integer r

(iii) $Q_k(s) = 0$ if $s < 0$

(iv) $Q_k(h) = \sum_{(v_1,\cdots v_k)\in P_{0(k-1),1(1)}} A_{v_1}\cdots A_{k},\ k\geq 1$

(v) $Q_1(jh) = A_j\,\mathrm{sgn}(\max\{0,\ 3-j\}),\ j\geq 0.$

(vi) $X^{(k)}(t_1^-, t_1), = (-A_0)^k = (-1)^k A_0^k,\ k = 1,2,\cdots$

(vii) $X^{(k)}(t_1^+, t_1), = 0$

(viii) $\Delta X(t_1 - jh, t_1) = \begin{cases} I_n, & \text{if } j = 0 \\ 0, & \text{otherwise} \end{cases}$

Proof

(i) $Q_1(0) = A_0Q_0(0) + A_1Q_0(-h) + A_2Q_0(-2h) = A_0I_n + A_1.0 + A_2.0 = A_0$

Assume that $Q_k(0) = A_0^k$, for $1 < k \le n$, for any integer n.

Then, $Q_{n+1}(0) = A_0Q_n(0) + A_1Q_n(-h) + A_2Q_n(-2h)$

$= A_0A_0^n + A_1Q_n(-h) + A_2Q_n(-2h)$, (by the induction hypothesis).

We need to prove that $Q_n(-h) = 0 = Q_n(-2h)$

Assertion: $Q_k(s) = 0$ if $s < 0$

Proof: $Q_1(s) = A_0Q_0(s) + A_1Q_0(s-h) + A_2Q_0(s-2h)$

$= A_0.0 + A_1.0 + A_2.0 = 0$, (by the definition of $Q_0(s)$).

So the assertion is true for $k = 1$.

Assume that $Q_k(s) = 0$ for $s < 0$, *for* $2 \le k \le n$, for some integer n. Then

$Q_{n+1}(s) = A_0Q_n(s) + A_1Q_n(s-h) + A_2Q_n(s-2h)$

$= A_0.0 + A_1.0 + A_2. = 0,$

by the induction hypothesis, since $s < 0, \Rightarrow s - h < 0, s - 2h < 0$.

Therefore, $Q_k(s) = 0 \;\; \forall s < 0$. Hence $Q_{n+1}(0) = A_0^{n+1}$, proving that $Q_k(0) = A_0^k$, for every nonnegative integer, k

(ii) Let $k = 1$ and let $s \neq rh$ for any integer r. Then

$Q_1(s) = A_0Q_0(s) + A_1Q_0(s-h) + A_2Q_0(s-2h)$

$= A_0.0 + A_1.0 + A_2.0 = 0$, since $s \notin \{0, h, 2h.\}$

Assume that $Q_k(s) = 0$ *for* $2 \le k \le n$, for some integer n. Then

$Q_{n+1}(s) = A_0Q_n(s) + A_1Q_n(s-h) + A_2Q_n(s-2h)$

$= A_0.0 + A_1.0 + A_2.0 = 0,$

by the induction hypothesis. Hence $Q_k(s) = 0 \;\forall s \neq rh$, for any integer r

(iii) This has already been proved.

(iv) $Q_1(h) = A_0Q_0(h) + A_1Q_0(0) + A_2Q_0(-h) = 0 + A_1 + 0 = A_1$, by the definition of Q_0

$= 0 + A_1 + 0 = A_1 = \sum_{v_1 \in P_{0(1-1),1(1)}} A_{v_1}$. So (iv) is true for $k = 1$.

Assume (iv) is true for $2 \le k \le n$, for some integer n. Then

$Q_{n+1}(h) = A_0Q_n(h) + A_1Q_n(0) + A_1Q_n(-h)$

$Q_n(h) = \sum_{(v_1, \ldots v_n) \in P_{0(n-1),1(1)}} A_{v_1} \cdots A_{v_n}$ by the induction hypothesis.

$Q_n(0) = A_0^n$, $Q_n(-h) = 0$, by (i) and (iii) respectively

Therefore, $Q_{n+1}(h) = \sum_{(v_1, \cdots, v_{n+1}) \in P_{0(n-1),1(1)}} A_{v_1} \cdots A_{v_{n+1}}$, with a leading A_0)

$+ \sum_{(v_1, \ldots v_{n+1}) \in P_{0(n),1(1)}} A_{v_1} \cdots A_{v_{n+1}}$, with a leading A_1

$= \sum_{(v_1, \ldots v_{n+1}) \in P_{0(n),1(1)}} A_{v_1} \cdots A_{v_{n+1}} = \sum_{(v_1, \cdots, v_{n+1}) \in P_{0(n+1-1),1(1)}} A_{v_1} \cdots A_{v_{n+1}} = \sum_{(v_1, \cdots, v_{n+1}) \in P_{0(n+1-1),1(1)}} A_{v_1} \cdots A_{v_{n+1}}$

So (iv) is true for $k = n + 1$, hence true for all $k \ge 1$.

(v) $Q_1(0) = A_0$, $Q_1(h) = A_1$, by (i) and (ii) respectively

$Q_1(2h) = A_0Q_0(2h) + A_1Q_0(h) + A_2Q_0(0)$

$= A_0.0 + A_1.0 + A_2 = A_2$

For $j \ge 3$, $Q_1(jh) = A_0Q_0(jh) + A_1Q_0([j-1]h) + A_2Q_0([j-2]h)$

Now $j > 0$, $j-1 > 0$ and $j-2 > 0$, since $j \geq 3$

Therefore, $Q_1(jh) = 0 \;\; \forall \; j \geq 3$ (by the definition of Q_0), proving (v).

(vi) $\quad X^{(k)}(t_1^-, t_1) = \frac{\partial}{\partial \tau} X^{(k-1)}(\tau, t_1)|_{\tau = t_1^-}$

For $k = 1$, this yields

$X^{(1)}(t_1^-, t_1) = -X(t_1^-, t_1)A_0 - X((t_1+h)^-, t_1)A_1 - X((t_1+2h)^-, t_1)A_2$

$= -A_0 - 0.A_1 - 0\; A_2$, since $t_1 + jh > t_1$ for $j = 1, \quad 2,$

Therefore $X^{(1)}(t_1^-, t_1) = -A_0 = (-1)A_0$. Note that for τ sufficiently, close to t_1, $\tau + h > t_1$

$$X^{(2)}(t_1^-, t_1) = -\left[\frac{\partial}{\partial \tau} X^{(1)}(\tau, t_1\right]|_{\tau = t_1^-}$$

$$= -X^{(1)}(t_1^-, t_1)A_0 - X^{(1)}((t_1+h)^-, t_1)A_1 - X^{(1)}((t_1+2h)^-, t_1)A_2 = -(-A_0)A_0$$
$$-[-X((t_1+h)^-, t_1)A_0 - X((t_1+2h)^-, t_1)A_1 - X((t_1+3h)^-, t_1)A_2]A_1$$
$$-[-X((t_1+2h)^-, t_1)A_0 - X((t_1+3h)^-, t_1)A_1 - X((t_1+4h)^-, t_1)A_2]A_2$$
$$= A_0^2 - [-0\,A_0 - 0\,A_1 - 0\,A_2]\,A_1 - [-0\,A_0 - 0\,A_1 - 0\,A_2]A_2 = A_0^2 = (-1)^2 A_0$$

Assume that $X^{(k)}(t_1^-, t_1) = (-1)^k A_0^k$, for $3 \leq k \leq n$, for some integer n.

Then $X^{(n+1)}(t_1^-, t_1) = \left[\frac{\partial}{\partial \tau} X^{(n)}(\tau, t_1)\right]\Big|_{\tau = t^-} = -X^{(n)}(t_1^-, t_1)A_0$

$-X^{(n)}((t_1+h)^-, t_1)A_1 - X^{(n)}((t_1+2h)^-, t_1)A_2 \qquad = (-1)(-1)^n A_0^n A_0 - 0A_1 - 0A_2 = (-1)^{n+1} A_0^{n+1}$.

Therefore $X^{(k)}(t_1^-, t_1) = (-1)^k A_0^k$, proving (vi)

(vii) $X^{(k)}(t_1^+, t_1) = \lim_{t_1 < \tau < t_1 + h} X^{(k)}(\tau, t_1) = 0$, since $\tau > t_1$. Therefore $X^{(k)}(t_1^+, t_1) = 0$, proving

(viii) $\quad \frac{\partial}{\partial \tau} X(\tau, t_1) = -X(\tau, t_1)A_0 - X(\tau + h, t_1)A_1 - X(\tau + 2h, t_1)A_2,$

for $0 < \tau < t_1, \tau \neq t_1 - jh; j = 0, 1, \cdots$, where $X(\tau, t) = \begin{cases} I_n, \; \tau = t \\ 0, \tau > t \end{cases}$

Let j be a non-negative number such that $t_1 - jh > 0$.

Then we integrate the system (3), apply the above initial matrix function condition and the fundamental theorem of calculus, (F.T.C.) to get:

$$\int_0^{(t_1 - jh)^-} \frac{\partial}{\partial \tau}[X(\tau, t_1)]d\tau = X(t_1 - jh)^-, t_1) - X(0, t_1), \text{ (by the F.T.C.)}$$
$$= -\int_0^{(t_1 - jh)^-} [X(\tau, t_1)A_0 + X(\tau + h, t_1)A_1 + X(\tau + 2h, t_1)A_2]d\tau$$

Similarly,

$$X((t_1 - jh)^+, t_1) - X(0, t_1)$$
$$= -\int_0^{(t_1 - jh)^+} [X(\tau, t_1)A_0 + X(\tau + h, t_1)A_1 + X(\tau + 2h, t_1)A_2]d_\tau$$

Therefore, $X((t_1 - jh)^-, t_1) - X((t_1 - jh)^+, t_1)$

$$= -\int_{(t_1 - jh)^-}^{(t_1 - jh)^+} [X(\tau, t_1)A_0 + X(\tau + h, t_1)A_1 + X(\tau + 2h, t_1)A_2]d_\tau = 0,$$

since $\tau -> X(\tau, t_1)A_0 + X(\tau + h, t_1)A_1 + X(\tau + 2h, t_1)A_2$ is bounded and integrable (being of bounded variation) and the fact that $\lim_{\varepsilon \to 0} \int_{a-\varepsilon}^{a+\varepsilon} f(t)\,dt = 0$

for any bounded integrable function, f. Therefore $\Delta X(t_1, -jh, t_1) \quad = 0$, for $j \neq 0$

For $j = 0$, we have $\Delta X(t_1, t_1) = X(t_1^-, t_1) - X(t_1^+, t_1) = I_n - 0 = I_n$, completing the proof of (viii). See Bounded variation (2012) for detailed discussion on functions of bounded variation.

2.6 Lemma on $Q_k(jh); j \in \{2k-2, 2k-1, \cdots\}, k \geq 1$

For $k \geq 1$

$$Q_k(jh) = \begin{cases} 0 \text{ if } j \geq 2k+1 & \text{(i)} \\ A_2^k \text{ if } j = 2k & \text{(ii)} \\ \sum_{(v_1\cdots v_k)\in P_{1(1),2(k-1)}} A_{v_1}\cdots A_{v_k} = \sum_{r=0}^{k-1} A_2^r A_1 A_2^{k-1-r}, \text{ if } j = 2k-1 & \text{(iii)} \\ \sum_{(v_1\cdots v_k)\in P_{1(2k-j),2(j-k)}} A_{v_1}\cdots A_{v_k} + \sum_{(v_1\cdots v_k)\in P_{0(1),2(k-1)}} A_{v_1}\cdots A_{v_k}, \text{ if } j = 2k-2 & \text{(iv)} \end{cases}$$

Proof

Note that the first summations in (iv), yield $\sum_{(v_1,\cdots,v_k)\in P_{1(2),2(k-2)}} A_{v_1}\cdots A_{v_k}$

$Q_k([2k+1]h) = A_0 Q_{k-1}([2k+1]h) + A_1 Q_{k-1}(2kh) + A_2 Q_{k-1}([2k-1]h)$

$k = 1 \Rightarrow Q_1(3h) = 0$, by lemma 1.4. Clearly $Q_1(jh) = 0$, for $j \geq 3$

So the lemma is valid for $j \geq 2k+1$, when $k = 1$

Assume that the lemma is valid for $j \geq 2k+1$, for every integer, k such that $2 \leq k \leq n$, for some integer n. Then

$$Q_n([2n+1]h) = A_0 Q_{n-1}([2n+1]h) + A_1 Q_{n-1}(2nh) + A_2 Q_{n-1}([2n-1]h) \quad (10)$$

$= A_0 . 0 + A_1 .0 + A_2 .0 = 0$ (by the induction hypothesis), since

$j = 2n+1 = 2(n-1)+3 > 2(n-1)+1$; $j-1 = 2n = 2(n-1)+2 > 2(n-1)+1$

$j-2 = 2n-1 = 2(n-1)+1$ and $n-1 < n$

Equivalently, on the right-hand side of (1.10) set

$k = n-1$, $j = 2n+1$, in the first term; $k = n-1$, $j = 2n$, in the second term

$n-1$, $j = 2n-1$, in the third term. Then clearly $k < n$ and $j \geq 2k+1$

Hence the induction hypothesis applies to the right-hand side of (1.10), yielding 0 in each term and consequently 0 for the sum of the terms. Therefore, $Q_n([2n+1]h) = 0$

For any $j > 2n+1$, $Q_n(jh) = A_0 Q_{n-1}(jh) + A_1 Q_{n-1}([j-1]h) + A_2 Q_{n-1}([j-2]h)$

Now $j > 2n+1 \Rightarrow j-2 > 2n+1-2 = 2(n-1)+1$ and $j-1 > 2(n-1)+1$.

Hence $Q_n(jh) = 0$, $\forall j > 2n+1$. Combine this with the case $j = 2n$, to conclude that

$Q_n(jh) = 0$, $\forall j \geq 2n+1$, proving that $Q_k(jh) = 0, \forall j \geq 2k+1$, $k \geq 1$, as required in (i) of lemma 2.5.

(ii) Consider $Q_k(2kh)$, for $k = 1$; this yields $Q_1(2h) = A_2$, by lemma 1.5.

So (ii) is valid for k =1.

Assume the validity of (ii) for $1 \leq k \leq n$, for some integer n. Then

$$Q_{n+1}(2[n+1]h) = A_0 Q_n(2[n+1]h) + A_1 Q_n([2n+1]h) + A_2 Q_n([2n]h)$$

$Q_n(2[n+1]h) = 0$, by (i), and $Q_n([2n+1]) = 0$, by (i) of lemma 2.5, since $(2[n+1]h) > 2n+1$, and $2n+1 = 2(n)+1$.

$Q_n(2nh) = A_2^n$, by the induction hypothesis; therefore, $A_2 Q_n(2nh) = A_2^{n+1}$

and $Q_{n+1}(2[n+1]h) = A_2^{n+1}$. Hence $Q_k(2kh) = A_2^k, \forall k \geq 1, k$ integer, proving

(iii) For k = 1, $Q_k(2[k-1]h) = Q_1(h) = A_1$, by lemma 1.5 .

Now $\sum_{(v_1,\cdots,v_k)\in P_{1(1),2(k-1)}} A_{v_1}\cdots A_{v_k} = \sum_{v_1\in P_{1(1)}} A_{v_1} = A_1$, for $k = 1$. So (iii) is valid for k = 1

Assume the validity of (iii) for $1 < k \leq n$, for some integer n. Then

$$Q_{n+1}([2(n+1)-1]h) = Q_{n+1}([2n+1]h)$$
$$= A_0 \; Q_n([2n+1]h) + A_1 Q_n(2nh) + A_2 Q_n([2n-1]h)$$

Now $Q_n([2n+1]h) = 0$ by (i), $Q_n(2nh) = A_2^n$ by (ii)

$Q_n([2n-1]h) = \sum_{(v_1\cdots v_k)\in P_{1(1),2(n-1)}} A_{v_1}\cdots A_{v_n}$, by the induction hypothesis .

Therefore,

$$Q_{n+1}([2(n+1)-1]h) = A_1A_2^n + A_2 \sum_{(v_1,\cdots,v_n)\in P_{1(1),2(n-1)}} A_{v_1}\cdots A_{v_n}$$

$$= A_1A_2^n + \sum_{(v_1,\cdots,v_{n+1})\in P_{1(1),2(n+1-1)}} A_{v_1}\cdots A_{v_{n+1}},$$

with leading A_2 in each permutation of the $A_{v_j}s$, $j=1,\cdots n+1$, in the above summation.

Since A_1 appears only once in each permutation it can only lead in one and only one permutation, in this case $A_1A_2^n$. In all other permutations A_1 will occupy positions 2, 3, … up the last position $n+1$. So the above expression for $Q_{n+1}([2(n+1)-1]h)$

is the same as:

$\sum A_{v_1}\cdots A_{v_{n+1}}$, proving that

$Q_k([2k-1]h) = \sum_{(v_1,\cdots,v_k)\in P_{1(1),2(k-1)}} A_{v_1}\cdots A_{v_k}$. So, the first part of (iii) is proved.

To prove the second part, note that $\sum_{r=0}^{k-1} A_2^r A_1 A_2^{k-1-r}$ is the sum of the permutations of A_1 and A_2

which A_1 appears once and A_2 appears $k-1$ times in each permutation.

In the first permutation, corresponding to $r = 0$, A_1 occupies the first postion (A_1 leads), …, in the last permutation, corresponding to $r = k - 1$, A_1 occupies the last position (A_1 trails).

the term under the summation represents the permutation in which A_1 occupies the $(r+1)^{st}$ position.

Thus

(iv) Consider $Q_k([2k-2]h)$, for $k=1$; this yields $Q_1(0) = A_0$ (by lemma 2.5).

Let us look at the right-hand side of (iv) in lemma 1.6.

If $k=1$, and $j=2k-2$, then $\sum_{(v_1,\ldots,v_k)\in P_{1(2),2(-1)}} A_{v_1}..A_{v_k} = \sum_{v_1\in P_{1(2),2(-1)}} A_{v_1} = 0$, by the summation infeasibility.

Now $\sum_{(v_1,\ldots,v_k)\in P_{0(1),2(k-1)}} A_{v_1}\ldots A_{v_k} = \sum_{v_1\in P_{0(1)}} A_{v_1} = A_0$, for $k=1$. So, (iv) is valid for $k=1$.

Assume the validity of (iv) for $1 < k \le n$ for some integer n. Then

$$Q_{n+1}(([2n+1]-2)h) = Q_{n+1}(2n)$$
$$= A_0Q_n(2nh) + A_1Q_n([2n-1]h) + A_2Q_n([2n-2]h).$$

$Q_n(2nh) = A_2^n$, by (ii)

$$Q_n([2n-1]h) = \sum_{(v_1,\cdots,v_n)\in P_{1(1),2(n-1)}} A_{v_1}\cdots A_{v_n} = \sum_{r=0}^{n-1} A_2^r A_1 A_2^{n-1-r}, \quad \text{by (iii)}.$$

$$Q_n([2n-2]h) = \sum_{(v_1,\cdots,v_n)\in P_{1(2n-[2n-2]),2(2n-2-n)}} A_{v_1}\cdots A_{v_n} + \sum_{(v_1,\cdots,v_n)\in P_{0(1),2(n-1)}} A_{v_1}\cdots A_{v_n},$$

(by the induction hypothesis)

$$= \sum_{(v_1,\cdots,v_n)\in P_{1(2),2(n-2)}} A_{v_1}\cdots A_{v_n} + \sum_{(v_1,\cdots,v_n)\in P_{0(1),2(n-1)}} A_{v_1}\cdots A_{v_n}.$$ Consequently,

$$Q_{n+1}(2nh) = A_0A_2^n + A_1 \sum_{(v_1,\cdots,v_n)\in P_{1(2),2(n-1)}} A_{v_1}\cdots A_{v_n}$$
$$+ A_2 \sum_{(v_1,\cdots,v_n)\in P_{1(1),2(n-2)}} A_{v_1}\cdots A_{v_n} + A_2 \sum_{(v_1,\cdots,v_n)\in P_{0(1),2(n-1)}} A_{v_1}\cdots A_{v_n}.$$

$$= A_0 A_2^n + \sum_{(v_1,\cdots,v_{n+1})\in P_{0(1),2(n)}} A_{v_1}\cdots A_{v_{n+1}} \text{ (with a leading } A_2)$$
$$+ \sum_{(v_1,\cdots,v_{n+1})\in P_{1(2),2(n-1)}} A_{v_1}\cdots A_{v_{n+1}} \text{ (with a leading } A_1)$$
$$+ \sum_{(v_1,\cdots,v_{n+1})\in P_{1(2),2(n-1)}} A_{v_1}\cdots A_{v_{n+1}} \text{ (with a leading } A_2)$$
$$= \sum_{(v_1,\cdots,v_{n+1})\in P_{0(1),2(n)}} A_{v_1}\cdots A_{v_{n+1}} + \sum_{(v_1,\cdots,v_{n+1})\in P_{1(2),2(n-1)}} A_{v_1}\cdots A_{v_{n+1}}$$

Notice that if $k = n+1$ and $j = 2(n+1)-2 = 2n,$ then $2k - j = 2$ and $j - k = n-1.$ So (iv) is proved for $k = n+1$, and hence (iv) is valid . This completes the proof of the lemma.
Lemma 2.6 can be restated in an equivalent form, devoid of explicit piece-wise representation as follows:

2.7 Lemma on $Q_k(jh);\ j \in \{2k-2, 2k-1, \cdots\}, k \geq 1$ using a composite function

For all nonnegative integers j and $k,$ such that $j \geq 2k-2, k \geq 1,$

$$Q_k(jh)$$
$$= \left[\sum_{(v_1,\cdots,v_k)\in P_{1(2k-j),2(j-k)}} A_{v_1}\cdots A_{v_k}\right]\operatorname{sgn}(\max\{0, 2k+1-j\})$$
$$+ \left[\sum_{(v_1,\cdots,v_k)\in P_{0(1),2(k-1)}} A_{v_1}\cdots A_{v_k}\right]\operatorname{sgn}(\max\{0,\ 2k-1-j\}).$$

Proof: If $j \geq 2k+1$, both signum functions vanish, proving (i) of lemma 2.6.
If $j = 2k$, the second signum vanishes and the first yields 1, proving (ii).
If $j = 2k-1$, the second signum vanishes and the first yields 1, proving (iii).
If $j = 2k-2$, both signum functions yields 1, proving (iv).

III. RESULTS AND DISCUSSIONS

3.1 Theorem on $Q_k(jh);\ 0 \leq j \leq k, k \neq 0$

For $0 \leq j \leq k,\ j,\ k$ integers, $k \neq 0,$

$$Q_k(jh) = \sum_{r=0}^{\left[\left[\frac{j}{2}\right]\right]} \sum_{(v_1,\cdots,v_k)\in P_{0(r+k-j),1(j-2r),2(r)}} A_{v_1}\cdots A_{v_k}$$

Proof

$$k \geq 1 \Rightarrow Q_k(0) = A_0^k,\ Q_k(h) = \sum_{(v_1,\cdots,v_k)\in P_{0(k-j),1(j)}} A_{v_1}\cdots A_{v_k}, \quad \text{(by lemma 2.5)}$$

$$j = 0 \Rightarrow r = 0 \Rightarrow \text{rhs} = \sum_{(v_1,\cdots,v_k)\in P_{0(0+k-0),1(0-0),2(0)}} A_{v_1}\cdots A_{v_k} = A_0^k$$

$$j = 1 \Rightarrow r = 0 \Rightarrow \text{rhs} = \sum_{(v_1,\cdots,v_k)\in P_{0(0+k-1),1(1-0),2(0)}} A_{v_1}\cdots A_{v_k} = \sum_{(v_1,\cdots,v_k)\in P_{0(k-1),1(1)}} A_{v_1}\cdots A_{v_k}$$

$$j=2, k\geq 2 \Rightarrow Q_2(2h)=A_0Q_1(2h)+A_1Q_1(h)+A_2Q_1(0)$$

$$=A_0A_2+A_1A_1+A_2A_0 \quad \text{(by lemma 2.5)}$$

$$j=2\Rightarrow r\in\{0,1\}\Rightarrow \text{rhs}=\sum_{(v_1,v_2)\in P_{0(0+0-0),1(2-0),2(0)}} A_{v_1}A_{v_2}+\sum_{(v_1,v_2)\in P_{0(1+2-2).1(2-2),2(1)}} A_{v_1}A_{v_2}$$

$$=A_1^2+A_0A_2+A_2A_0$$

So, the theorem is true for $j\in\{0, 1\}$, $k\geq 1$ and for $j=2=k$.

Assume that the theorem is valid for all triple pairs $\tilde{j},\tilde{k},Q_{\tilde{k}}(\tilde{j}h); j,k,Q_k(jh)$ for which $\tilde{j}+\tilde{k}\leq j+k$, for some $j,k: k\geq j\geq 3$. Then

$Q_{k+1}(jh)=A_0Q_k(jh)+A_1Q_k([j-1]h)+A_2Q_k([j-2]h)$

Now, $j\leq k+1 \Rightarrow j-1\leq k$ and $j-2\leq k-1<k$. So, we may apply the induction hypothesis to the right-hand side of $Q_{k+1}(jh)$ to get:

$$Q_{k+1}(jh) = A_0\sum_{r=0}^{\left[\left[\frac{j}{2}\right]\right]}\sum_{(v_1,\cdots,v_k)\in P_{0(r+k-j),1(j-2r),2(r)}} A_{v_1}\cdots A_{v_k} \tag{11}$$

$$+A_1\sum_{r=0}^{\left[\left[\frac{j-1}{2}\right]\right]}\sum_{(v_1,\cdots,v_k)\in P_{0(r+k-(j-1)),1(j-1-2r),2(r)}} A_{v_1}\cdots A_{v_k} \tag{12}$$

$$+A_2\sum_{r=0}^{\left[\left[\frac{j-2}{2}\right]\right]}\sum_{(v_1,\cdots,v_k)\in P_{0(r+k-(j-2)),1(j-2-2r),2(r)}} A_{v_1}\cdots A_{v_k} \tag{13}$$

Two cases arise: j even and j odd

Case 1: j even. Then $j-1$ is odd are $j-2$ is even; thus

$$\left[\left[\frac{j-1}{2}\right]\right]=\left[\left[\frac{j-2}{2}\right]\right]=\frac{j}{2}-1 \text{ and } \left[\left[\frac{j}{2}\right]\right]=\frac{j}{2}$$

The summations in (11) are all feasible, since $j\geq 2r$, noting that $r\in\left\{1,2,\cdots,\frac{j}{2}\right\}$.

So the right hand side of (11) can be rewritten as:

$$\sum_{r=0}^{\left[\left[\frac{j}{2}\right]\right]}\sum_{(v_1,\cdots,v_{k+1})\in P_{0(r+(k+1)-j),1(j-2r),2(r)}} A_{v_1}\cdots A_{v_k}, \tag{14}$$

with a leading A_0

(12) can be rewritten in the form:

$$A_1\sum_{r=0}^{\frac{j}{2}-1}\sum_{(v_1,\cdots,v_k)\in P_{0(r+k-(j-1)),1(j-1-2r),2(r)}} A_{v_1}\cdots A_{v_k} \tag{15}$$

We need to incorporate $\frac{j}{2}$ in the range of r. If $r=\frac{j}{2}$, then $j-1-2r=j-1-j=-1$.

Therefore, the summation $\sum_{(v_1,\cdots,v_k)\in P_{0(k+1-\frac{j}{2})),1(-1),2(\frac{j}{2})}} A_{v_1}\cdots A_{v_k}$ is infeasible; hence it is set equal to 0. Thus the case $r=\dfrac{j}{2}$ may be included in the expression (2.5) to yield:

$$A_1\sum_{r=0}^{\frac{j}{2}}\sum_{(v_1,\cdots,v_k)\in P_{0(r+(k+1)-j)),1(j-1-2r),2(r)}} A_{v_1}\cdots A_{v_k}=\sum_{r=0}^{\frac{j}{2}}\sum_{(v_1,\cdots,v_{k+1})\in P_{0(r+(k+1)-j)),1(j-1-2r),2(r)}} A_{v_1}\cdots A_{v_{k+1}}, \qquad (16)$$

with a leading A_1

(2.3) may be rewritten in the form:

$$A_2\sum_{r=0}^{\frac{j}{2}-1}\sum_{(v_1,\cdots,v_k)\in P_{0(r+1+(k+1)-j)),1(j-2(r+1)),2(r)}} A_{v_1}\cdots A_{v_k} \qquad (17)$$

If $r=\dfrac{j}{2}$, then $j-2(r+1)=j-j-2=-2$; so the summations with $r=\dfrac{j}{2}$, may be set equal to 0, being infeasible, yielding:

$$A_2\sum_{r=0}^{\frac{j}{2}}\sum_{(v_1,\cdots,v_k)\in P_{0(r+1+(k+1)-j)),1(j-2(r+1)),2(r)}} A_{v_1}\cdots A_{v_k}=A_2\sum_{r=1}^{\frac{j}{2}+1}\sum_{(v_1,\cdots,v_k)\in P_{0(r+(k+1)-j)),1(j-2r)),2(r-1)}} A_{v_1}\cdots A_{v_k} \qquad (18)$$

(We used the change of variables technique: $r\to r-1$ in the summand, $r\to r+1$ in the limits).
If $r=\frac{j}{2}+1$, then $j-2r=j-j-2=-2$; so the summations with $r=\frac{j}{2}+1$ may be equated to 0 and dropped.
If $r=0$, then $r-1=-1$. Therefore the summations with
$r=0$ are infeasible and hence set equal to 0. Thus (18) is the same as:

$$A_2\sum_{r=0}^{\frac{j}{2}}\sum_{(v_1,\cdots,v_k)\in P_{0(r+(k+1)-j)),1(j-2r)),2(r-1)}} A_{v_1}\cdots A_{v_k}=\sum_{r=0}^{\frac{j}{2}}\sum_{(v_1,\cdots,v_k)\in P_{0(r+(k+1)-j)),1(j-2r)),2(r)}} A_{v_1}\cdots A_{v_k}, \qquad (19)$$

with a leading A_2.

Therefore $Q_{k+1}(jh)$

$$=\sum_{r=0}^{\left[\left[\frac{j}{2}\right]\right]}\sum_{(v_1,\cdots,v_{k+1})\in P_{0(r+(k+1)-j),1(j-2r),2(r)}} A_{v_1}\cdots A_{v_{k+1}},\quad \text{with a leading } A_0$$

$$+\sum_{r=0}^{\left[\left[\frac{j}{2}\right]\right]}\sum_{(v_1,\cdots,v_{k+1})\in P_{0(r+(k+1)-j),1(j-2r),2(r)}} A_{v_1}\cdots A_{v_{k+1}},\quad \text{with a leading } A_1$$

$$+\sum_{r=0}^{\left[\left[\frac{j}{2}\right]\right]}\sum_{(v_1,\cdots,v_{k+1})\in P_{0(r+(k+1)-j),1(j-2r),2(r)}} A_{v_1}\cdots A_{v_{k+1}},\quad \text{with a leading } A_2$$

$$=\sum_{r=0}^{\frac{j}{2}}\sum_{(v_1,\cdots,v_{k+1})\in P_{0(r+(k+1)-j),1(j-2r),2(r)}} A_{v_1}\cdots A_{v_{k+1}}=\sum_{r=0}^{\left[\left[\frac{j}{2}\right]\right]}\sum_{(v_1,\cdots,v_{k+1})\in P_{0(r+(k+1)-j),1(j-2r),2(r)}} A_{v_1}\cdots A_{v_{k+1}}$$

This concludes the proof of the theorem for j even.
If $k = 0$, then $j = 0$ since $0\le j\le k$, yielding $Q_0(0) = I_n$, the $n\times n$ identity.
Case 2: j odd. Then $j-1$ is even, $j-2$, is odd and $j-3$ is even. Hence

$$\left[\left[\frac{1}{2}(j-1)\right]\right]=\frac{1}{2}(j-1)=\left[\left[\frac{1}{2}j\right]\right],\ \text{and}\ \left[\left[\frac{1}{2}(j-2)\right]\right]=\left[\left[\frac{1}{2}(j-3)\right]\right]=\frac{1}{2}(j-3)=\left[\left[\frac{1}{2}j\right]\right]-1$$ A

gain (11) is the same as:

$$\sum_{r=0}^{\left[\left[\frac{j}{2}\right]\right]}\sum_{(v_1,\cdots,v_{k+1})\in P_{0(r+(k+1)-j),1(j-2r),2(r)}} A_{v_1}\cdots A_{v_{k+1}},\ (\text{with a leading } A_0) \tag{20}$$

(2.5) is the same as:

$$A_1\sum_{r=0}^{\left[\left[\frac{j}{2}\right]\right]}\sum_{(v_1,\cdots,v_{k+1})\in P_{0(r+(k+1)-j),1(j-1-2r),2(r)}} A_{v_1}\cdots A_{v_k}=\sum_{r=0}^{\left[\left[\frac{j}{2}\right]\right]}\sum_{(v_1,\cdots,v_{k+1})\in P_{0(r+(k+1)-j),1(j-2r),2(r)}} A_{v_1}\cdots A_{v_{k+1}}, \tag{21}$$

with a leading A_1, since $r+(k+1)-j,\ j-1-2r$ and r are all nonnegative for

$$r\in\left\{0,1,\cdots,\left[\left[\frac{1}{2}j\right]\right]\right\}=\left\{0,1,\cdots,\left[\left[\frac{1}{2}(j-1)\right]\right]\right\}.$$

(2.7) can be rewritten in the form:

$$A_2\sum_{r=0}^{\left[\left[\frac{j}{2}\right]\right]-1}\sum_{(v_1,\cdots,v_k)\in P_{0(r+1+(k+1)-j),1(j-2(r+1)),2(r)}} A_{v_1}\cdots A_{v_k}$$

$$=A_2\sum_{r=1}^{\left[\left[\frac{j}{2}\right]\right]}\sum_{(v_1,\cdots,v_k)\in P_{0(r+1+(k+1)-j),1(j-2r)),2(r-1)}} A_{v_1}\cdots A_{v_k} \tag{22}$$

If $r = 0$, then $r - 1 = -1 < 0$. Therefore, the summations with $r = 0$ vanish, with (2.12) transforming to:

$$A_2 \sum_{r=0}^{\left[\left[\frac{j}{2}\right]\right]} \sum_{(v_1,\cdots,v_k)\in P_{0(r+1+(k+1)-j),1(j-2r),2(r-1)}} A_{v_1}\cdots A_{v_k}$$

$$= \sum_{r=1}^{\left[\left[\frac{j}{2}\right]\right]} \sum_{(v_1,\cdots,v_{k+1})\in P_{0(r+1+(k+1)-j),1(j-2r)),2(r)}} A_{v_1}\cdots A_{v_{k+1}}, \qquad (23)$$

with leading A_2.

Finally, $Q_{k+1}(jh)$ = (20) + (21) + (23), the same expression in each summation, but with leading A_0, A_1 and A_2 respectively. Consequently,

$$Q_{k+1}(jh) = \sum_{r=0}^{\left[\left[\frac{j}{2}\right]\right]} \sum_{(v_1,\cdots,v_{k+1})\in P_{0(r+(k+1)-j),1(j-2r),2(r)}} A_{v_1}\cdots A_{v_{k+1}},$$

completing the proof of the theorem for j odd. Hence the theorem has been proved for both cases; therefore, the validity of the theorem is established.

3.2 Theorem on $Q_k(jh);\ j \ge k \ge 1$

For $j \ge k \ge 1$, j, k integers,

$$Q_k(jh) = \begin{cases} \sum_{r=0}^{\left[\left[\frac{2k-j}{2}\right]\right]} \sum_{(v_1,\cdots,v_k)\in P_{0(r),1(2k-j-2r),2(r+j-k)}} A_{v_1}\cdots A_{v_k}, & 1 \le j \le 2k \\ 0, & j \ge 2k+1 \end{cases}$$

Proof

Consider $Q_k(jh)$, for $j \ge k \ge 1$.

For $k = 1$, we appeal to lemma 1.4 to obtain $Q_k(jh) = Q_1(jh) = \begin{cases} A_1, \text{ if } j = 1 \\ A_2, \text{ if } j = 2 \\ 0, \text{ if } j \ge 3 \end{cases}$

Hence, $Q_1(jh) = A_j \operatorname{sgn}(\max\{0,\ 3-j\}),\ j \ge 1.$

If $j = 1$, then $\frac{2k-j}{2} = \frac{1}{2}$; so $r = 0$ and the rhs summation $= A_1$.

If $j = 2$, then $\frac{2k-j}{2} = 0$; so $r = 0$, and the rhs summation $= A_2$. If $j \ge 3$, then $\frac{2k-j}{2} \le -\frac{1}{2}$; so r is infeasible $\Rightarrow$ the rhs summation $= 0$, for $j \ge 3$.

Therefore, in the stated formula, $Q_1(jh) = A_j \operatorname{sgn}(\max\{0,\ 3-j\})$, in agreement with lemma 2.5. Therefore the theorem is valid for $k = 1,\ j \ge k$.

Assume that the theorem is valid for $1 < k \le n \le j$, for some integer n. Then, for $j \ge n+1$,

$$Q_{n+1}(jh) = A_0 Q_n(jh) + A_1 Q_n([j-1]h) + A_2 Q_n([j-2]h).$$

We may apply the induction hypothesis to $Q_n(jh)$ to get

$$Q_n(jh) = \sum_{r=0}^{\left[\left[\frac{2n-j}{2}\right]\right]} \sum_{(v_1,\cdots,v_n)\in P_{0(r),1(2n-j-2r),2(r+j-n)}} A_{v_1}\cdots A_{v_n}, \text{ since } j \ge n.$$

Now, $j \ge n+1 \Rightarrow j-1 \ge n$, or $n \le j-1$. So, we may apply the induction principle to $Q_n([j-1]h)$ to get

$$Q_n([j-1]h) = \sum_{r=0}^{\left[\left[\frac{2n-[j-1]}{2}\right]\right]} \sum_{(v_1,\cdots,v_n)\in P_{0(r),1(2n-[j-1]-2r),2(r+[j-1]-n)}} A_{v_1}\cdots A_{v_n},$$

where all permutations are feasible. If $j-2 \ge n$, apply the induction hypothesis to $Q_n([j-2]h)$, to get

$$Q_n([j-2]h) = \sum_{r=0}^{\left[\left[\frac{2n-[j-2]}{2}\right]\right]} \sum_{(v_1,\cdots,v_n)\in P_{0(r),1(2n-[j-2]-2r),2(r+[j-2]-n)}} A_{v_1}\cdots A_{v_n}.$$

Hence, $Q_{n+1}(jh)$

$$= A_0 \sum_{r=0}^{\left[\left[\frac{2n-j}{2}\right]\right]} \sum_{(v_1,\cdots,v_n)\in P_{0(r),1(2n-j-2r),2(r+j-n)}} A_{v_1}\cdots A_{v_n} + A_1 \sum_{r=0}^{\left[\left[\frac{2n-[j-1]}{2}\right]\right]} \sum_{(v_1,\cdots,v_n)\in P_{0(r),1(2n-[j-1]-2r),2(r+[j-1]-n)}} A_{v_1}\cdots A_{v_n}$$

$$+ A_2 \sum_{r=0}^{\left[\left[\frac{2n-[j-2]}{2}\right]\right]} \sum_{(v_1,\cdots,v_n)\in P_{0(r),1(2n-[j-2]-2r),2(r+[j-2]-n)}} A_{v_1}\cdots A_{v_n}$$

Case: j even. Then $2n-j$ is even. So $\left[\left[\frac{1}{2}(2n-j)\right]\right] = n-\frac{j}{2}$; $2n-(j-2)$ is even, so $\left[\left[\frac{1}{2}(2n-[j-2])\right]\right] = n-\frac{j}{2} = n+1-j$

$2n-[j-1]$ is odd. So

$$\left[\left[\frac{1}{2}(2n-[j-1])\right]\right] = \left[\left[\frac{1}{2}(2n-[j-1]-1)\right]\right] = n-\frac{j}{2}.$$

$2(n+1)-j$ is even; so $\left[\left[\frac{1}{2}(2[n+1]-j)\right]\right] = n+1-\frac{j}{2}$. Hence:

$$Q_{n+1}(jh) = A_0 \sum_{r=0}^{n-\frac{j}{2}} \sum_{(v_1,\cdots,v_n)\in P_{0(r),1(2n-j-2r),2(r+j-n)}} A_{v_1}\cdots A_{v_n} \tag{24}$$

$$+ A_1 \sum_{r=0}^{n-\frac{j}{2}} \sum_{(v_1,\cdots,v_n)\in P_{0(r),1(2n+1-j-2r),2(r+j-1-n)}} A_{v_1}\cdots A_{v_n} \tag{25}$$

$$+ A_2 \sum_{r=0}^{n+1-\frac{j}{2}} \sum_{(v_1,\cdots,v_n)\in P_{0(r),1(2[n+1]-j-2r),2(r+j-[n+1]-1)}} A_{v_1}\cdots A_{v_n} \tag{26}$$

Use the change of variables $\tilde{r} = r+1,$ in (2.14) to get

$$A_0 \sum_{\tilde{r}=1}^{1+n-\frac{j}{2}} \sum_{(v_1,\cdots,v_n)\in P_{0(\tilde{r}-1),1(2n-j-2[\tilde{r}-1]),2(\tilde{r}-1+j-n)}} A_{v_1}\cdots A_{v_n} = A_0 \sum_{r=1}^{1+n-\frac{j}{2}} \sum_{(v_1,\cdots,v_n)\in P_{0(r-1),1(2[n+1]-j-2r),2(r+j-[n+1])}} A_{v_1}\cdots A_{v_n}$$

$$= A_0 \sum_{r=0}^{1+n-\frac{j}{2}} \sum_{(v_1,\cdots,v_n)\in P_{0(r-1),1(2[n+1]-j-2r),2(r+j-[n+1])}} A_{v_1}\cdots A_{v_n},$$

(since the summation with $r=0$ is infeasible and hence equals 0).

$$= \sum_{r=0}^{\left[\left[\frac{2(n+1)-j}{2}\right]\right]} \sum_{(v_1,\cdots,v_{n+1})\in P_{0(r),1(2[n+1]-j-2r),2(r+j-[n+1])}} A_{v_1}\cdots A_{v_{n+1}}, \text{ with a leading } A_0. \qquad (27)$$

If we set $r = n+1-\frac{j}{2}$, in (25), then $2n+1-j-2r = 2n+1-j-2n-2+j = -1;$ so the summations with $r = n+1-\frac{j}{2}$ vanish, being infeasible. Therefore (2.15) is the same expression as:

$$A_1 \sum_{r=0}^{n+1-\frac{j}{2}} \sum_{(v_1,\cdots,v_n)\in P_{0(r),1(2n+1-j-2r),2(r+j-[n+1])}} A_{v_1}\cdots A_{v_n}$$

$$= \sum_{r=0}^{\left[\left[\frac{2(n+1)-j}{2}\right]\right]} \sum_{(v_1,\cdots,v_{n+1})\in P_{0(r),1(2n+1-j-2r),2(r+j-[n+1])}} A_{v_1}\cdots A_{v_{n+1}}, \qquad (28)$$

with a leading A_1.

Clearly (2.16) is the same expression as:

$$\sum_{r=0}^{\left[\left[\frac{2(n+1)-j}{2}\right]\right]} \sum_{(v_1,\cdots,v_{n+1})\in P_{0(r),1(2n+1-j-2r),2(r+j-[n+1])}} A_{v_1}\cdots A_{v_{n+1}}, \qquad (29)$$

with a leading A_2.

Add up (27), (28) and (29) to obtain:

$$Q_{n+1}(jh) = \sum_{r=0}^{\left[\left[\frac{2(n+1)-j}{2}\right]\right]} \sum_{(v_1,\cdots,v_{n+1})\in P_{0(r),1(2n+1-j-2r),2(r+j-[n+1])}} A_{v_1}\cdots A_{v_{n+1}}.$$

Hence, the theorem is valid for all $j \geq n+1;$ this completes the proof for the case j even.

Now consider the case: j odd. Then $2n - j$ is odd. Therefore, $\left[\left[\frac{1}{2}(2n-j)\right]\right] = \left[\left[\frac{1}{2}(2n-j-1)\right]\right] = n - \frac{1}{2}(j+1),$

$$2n-(j-2) \text{ is odd; so, } \left[\left[\frac{1}{2}(2n-(j-2))\right]\right] = \left[\left[\frac{1}{2}(2n-(j-2)-1)\right]\right] = n+1-\frac{1}{2}(j+1)$$

$$\left[\left[\frac{1}{2}(2n-j)\right]\right] = \left[\left[\frac{1}{2}(2n-j-1)\right]\right] = n-\frac{1}{2}(j+1). \text{ Clearly, } 2n-(j-1) \text{ is even;}$$

$$\text{so, } \left[\left[\frac{1}{2}(2n-(j-1))\right]\right] = \frac{1}{2}(2n-(j-1)) = \frac{1}{2}(2[n+1]-1-j) = n+1-\frac{1}{2}(j+1)$$

$$2(n+1)-j \text{ is odd; so, } \left[\left[\frac{1}{2}(2[n+1]-j)\right]\right] = \left[\left[\frac{1}{2}(2[n+1]-j-1)\right]\right] = n+1-\frac{1}{2}(j+1).$$

Hence: $Q_{n+1}(jh)$

$$= A_0 \sum_{r=0}^{n-\frac{(j+1)}{2}} \sum_{(v_1,\cdots,v_n)\in P_{0(r),1(2n-j-2r),2(r+j-n)}} A_{v_1}\cdots A_{v_n} \quad (30)$$

$$+ A_1 \sum_{r=0}^{n+1-\frac{(j+1)}{2}} \sum_{(v_1,\cdots,v_n)\in P_{0(r),1(2n+1-j-2r),2(r+j-1-n)}} A_{v_1}\cdots A_{v_n} \quad (31)$$

$$+ A_2 \sum_{r=0}^{n+1-\frac{(j+1)}{2}} \sum_{(v_1,\cdots,v_n)\in P_{0(r),1(2[n+1]-j-2r),2(r+j-[n+1]-1)}} A_{v_1}\cdots A_{v_n} \quad (32)$$

Note that $n+1-\frac{1}{2}(j+1) = \left[\left[\frac{(2(n+1)-j)}{2}\right]\right]$, as earlier established .Therefore using the change of variables $\tilde{r} = r+1$,in (30),we see that (30) is exactly the same expression as

$$A_0 \sum_{r=0}^{n+1-\frac{1}{2}(j+1)} \sum_{(v_1,\cdots,v_n)\in P_{0(r-1),1(2[n+1]-j-2r),2(r+j-[n+1])}} A_{v_1}\cdots A_{v_n}$$

$$= \sum_{r=0}^{\left[\left[\frac{2(n+1)-j}{2}\right]\right]} \sum_{(v_1,\cdots,v_{n+1})\in P_{0(r),1(2[n+1]-j-2r),2(r+j-[n+1])}} A_{v_1}\cdots A_{v_{n+1}}, \text{ with a leading } A_0. \quad (33)$$

(31) is exactly the same expression as:

$$\sum_{r=0}^{\left[\left[\frac{2(n+1)-j}{2}\right]\right]} \sum_{(v_1,\cdots,v_{n+1})\in P_{0(r),1(2[n+1]-j-2r),2(r+j-[n+1])}} A_{v_1}\cdots A_{v_{n+1}}, \text{ with a leading } A_1. \quad (34)$$

(32) is exactly the same expression as:

$$\sum_{r=0}^{\left[\left[\frac{2(n+1)-j}{2}\right]\right]} \sum_{(v_1,\cdots,v_{n+1})\in P_{0(r),1(2[n+1]-j-2r),2(r+j-[n+1])}} A_{v_1}\cdots A_{v_{n+1}}, \text{ with a leading } A_2. \quad (35)$$

Add up (33), (34) and (35) to obtain:

$$Q_{n+1}(jh) = \sum_{r=0}^{\left[\left[\frac{2(n+1)-j}{2}\right]\right]} \sum_{(v_1,\cdots,v_{n+1})\in P_{0(r),1(2[n+1]-j-2r),2(r+j-[n+1])}} A_{v_1}\cdots A_{v_{n+1}}, \quad (36)$$

proving the theorem for j odd, for the contingency $j-2 \geq n.$

Last case: $j-2 < n$. Then $j < n+2;$ but $j \geq n+1,$ forcing $j = n+1.$ We invoke theorem 3.1 to conclude that

$$Q_{n+1}([n+1]h) = \sum_{r=0}^{\left[\left[\frac{n+1)}{2}\right]\right]} \sum_{(v_1,\cdots,v_{n+1})\in P_{0(r+n+1-(n+1)),1(n+1-2r),2(r)}} A_{v_1}\cdots A_{v_{n+1}}.$$

Now set $j = n+1$, in the expression for $Q_{n+1}(jh)$, in theorem 3.2, to get

$$Q_{n+1}([n+1]h) = \sum_{r=0}^{\left[\left[\frac{n+1)}{2}\right]\right]} \sum_{(v_1,\cdots,v_{n+1})\in P_{0(r)),1(n+1-2r),2(r)}} A_{v_1}\cdots A_{v_{n+1}},$$

exactly the same expression as in theorem 3.1. This completes the proof of theorem 3.2.

Remarks

The expressions for $Q_k(jh)$ in theorems 3.1 and 3.2 coincide when $j = k \neq 0,$ as should be expected.

IV. CONCLUSION

The results in this article bear eloquent testimony to the fact that we have comprehensively extended the previous single-delay result by Ukwu (1992) together with appropriate embellishments through the unfolding of intricate inter–play of the greatest integer function and the permutation objects in the course of deriving the expressions for the determining matrices.By using the greatest integer function analysis, change of variables technique and deft application of mathematical induction principles we were able to obtain the structure of the determining matrices for the double–delay control model, without which the computational investigation of Euclidean controllability would be impossible. The mathematical icing on the cake was our deft application of the max and sgn functions and their composite function sgn (max {.,.}) in the expressions for determining matrices. Such applications are optimal, in the sense that they obviate the need for explicit piece–wise representations of those and many other discrete mathematical objects and some others in the continuum.

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