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Analyses of a mixing problem and associated delay models

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ABSTRACT: *In this article, we gave an exposition on a class of mixing problems as they relate to scalar delay differential equations. In the sequel we formulated and proved theorems on feasibility and forms of solutions for such problems, in furtherance of our quest to enhance the understanding and appreciation of delay differential equations and associated problems. We obtained our results using the method of steps and forward continuation recursive procedure.*

KEYWORDS: *Delay, Feasibility, Mixing, Problems, Solutions.*

I. INTRODUCTION

Dilution models are well known in the literature on ordinary differential equations. However literature on the extension of these models to delay differential equations is quite sparse and the associated analyses not thorough, detailed or general for the most part. See Driver (1977) for an example. This article leverages on the model in Driver to conduct detailed analyses of ordinary and associated delay differential equations models of mixing problems, with accompanying theorems, and corollary, together with appropriate feasibility conditions on the solutions.

II. PRELIMINARY DEFINITION

A linear delay differential system is a system of the form:

$$\dot{x}(t) = A(t)x(t) + B(t)x(t-h) + g(t), \tag{1}$$

where *A* and *B* are $n \times n$ matrix-valued functions on $\mathbf{R}$ and $h > 0$ is some constant and $g(t)$ is continuous.

Remarks

Any other appropriate conditions that could be imposed on g, *A* and *B* to guarantee existence of solution will still do. The need for appropriate specification of initial data will be looked at very shortly.
If $g(t) = 0 \ \forall t \in IR$, then (1) is called homogeneous. If *A* and *B* are time-independent, the system (1) is referred to as an autonomous delay system.

III. AN ORDINARY DIFFERENTIAL EQUATION DILUTION MODEL (MIXING PROBLEM)

Consider the following problem:

A G-gallon tank initially contains S_0 pounds of salt dissolved in W_0 gallons of water. Suppose that b_1 gallons of brine containing s_0 pounds of dissolved salt per gallon runs into the tank every minute and that the mixture (kept uniform by stirring) runs out of the tank at the rate of b_2 gallons per minute.

Note: We assume continual instantaneous, perfect mixing throughout the tank.
The following questions are reasonable:

a) Set up a differential equation for the amount of salt in the tank after t minutes.
1) What is the condition for the tankto overflow?
2) Not to overflow?
3) how much salt will be in the tank at the instant it begins to overflow?

b) Will the tank ever be empty?

Solution

Let $S(t)$ be the amount of salt in the tank at time t. Then b_1 gallon of brine flow into the tank every minute and each gallon contains *s* pounds of salt. Thus $b_1 s$ pounds of salt flow into the tank each minute.

Amount of salt flowing out of the tank every minute: at time t we have $S(t)$ lbs of salt and $W_0 + (b_1 - b_2)t$ gallons of solution in the tank, since there is a net increase of $(b_1 - b_2)$ gallons of solution every minute. Therefore, the salt concentration in the solution at time t is $\frac{S(t)}{W_0 + (b_1 - b_2}$ lbs per gallon, and salt leaves the tank at the rate

$$\left[\frac{S(t)}{W_0 + (b_1 - b_2)t} \text{ lbs/gallon}\right]\left[b_2 \text{ gallons/minute}\right] = \frac{b_2\, S(t)}{W_0 + (b_1 - b_2)t} \text{lbs/ min}.$$

Hence the net rate of change, $\frac{dS}{dt}$ of salt in the tank is given by

Net rate of change = salt inflow per minute - salt outflow per minute, expressed as:

$$\frac{dS}{dt} = b_1\, s_0 - \frac{b_2 S}{W_0 + (b_1 - b_2)t} \tag{2}$$

(2) $\Rightarrow \frac{dS}{dt} + \frac{b_2 S}{W_0 + (b_1 - b_2)t} = b_1\, s_0$, a first order differential equation $\frac{dS}{dt} + p(t)S = q(t)$, with

$$p(t) = \frac{b_2}{W_0 + (b_1 - b_2)t} \quad \text{and} \quad q(t) = b_1\, s_0$$

The integrating factor is

$$I(t) = e^{\int p(t)dt} = e^{\int \frac{b_2}{W_0 + (b_1 - b_2)t}dt} = e^{\frac{b_2}{b_1 - b_2}\text{Ln}\left|W_0 + (b_1 - b_0)t\right|}, \; b_1 \neq b_2$$

$$= \left|W_0 + (b_1 - b_2)t\right|^{\frac{b_2}{b_1 - b_2}}$$

Now $W_0 + (b_1 - b_2)t > 0$ makes practical sense, as the amount of brine cannot be negative or 0 at any time t.

Case i: $b_1 > b_2 \Rightarrow$ tank overflows at some time *t* if mixing is continual, thus

$$I(t) = \left(W_0 + (b_1 - b_2)t\right)^{\frac{b_2}{b_1 - b_2}}$$

The general solution given by, $S(t) = \frac{1}{I(t)}\left[\int q(t)I(t)dt + C\right]$, where C is an arbitrary constant.

Hence,

$$S(t) = \frac{1}{\left(W_0 + (b_1 - b_2)t\right)^{\frac{b_2}{b_1 - b_2}}}\left[\int b_1\, s_0 \left(W_0 + (b_1 - b_2)t\right)^{\frac{b_2}{b_1 - b_2}dt} + C\right]$$

$$= \frac{b_1 s_0}{[W_0 + (b_1 - b_2)t]^{\frac{b_2}{b_1-b_2}}} \left[\int [W_0 + (b_1 - b_2)t]^{\frac{b_2}{b_1-b_2}} \, dt + C \right]$$

$$= \frac{b_1 \ s_0}{[W_0 + (b_1 - b_2)t]^{\frac{b_2}{b_1-b_2}}} \left[\frac{1}{(b_1 - b_2)} \cdot \frac{1}{\left(\frac{b_2}{b_1 - b_2} + 1 \right)} [W_0 + (b_1 - b_2)t]^{\frac{b_2}{b_1-b_2}} + C \right]$$

$$= \frac{b_1 \ s_0}{[W_0 + (b_1 - b_2)t]^{\frac{b_2}{b_1-b_2}}} \left[\frac{1}{b_1} (W_0 + (b_1 - b_2)t)^{\frac{b_1}{b_1-b_2}} + C \right]. \qquad (3)$$

At the instant the tank overflows, $W_0 + (b_1 - b_2)t \quad = G$, so that $t = \dfrac{G - W_0}{b_1 - b_2}$

The amount of salt in the tank at that instant is $S\left(\dfrac{G - w_0}{b_1 - b_2} \right)$. Now, C can be obtained by noting that $S(0) = S_0$, yielding:

$$\frac{b_1 s_0}{W_0^{\frac{b_2}{b_1-b_2}}} \left[\frac{1}{b_1} W_0^{\frac{b_2}{b_1-b_2}} + C \right] = S_0, \ \Rightarrow C = \frac{s_0}{b_1 s_0} W_0^{\frac{b_2}{b_1-b_2}} - \frac{1}{b_1} W_0^{\frac{b_1}{b_1-b_2}}$$

Therefore,

$$S\left(\frac{G - W_o}{b_1 - b_2} \right) = \frac{b_1 s_0}{G^{\frac{b_2}{b_1-b_2}}} \left[\frac{1}{b_1} G^{\frac{b_1}{b_1-b_2}} + \frac{S_0}{b_1 s_0} W_0^{\frac{b_2}{b_1-b_2}} - \frac{1}{b_1} W_0^{\frac{b_1}{b_1-b_2}} \right]$$

$$= \frac{s_0}{G^{\frac{b_2}{b_1-b_2}}} \left[G^{\frac{b_1}{b_1-b_2}} + \frac{1}{s_0} S_0 W_0^{\frac{b_2}{b_1-b_2}} - W_0^{\frac{b_1}{b_1-b_2}} \right]$$

Case ii: $b_1 < b_2 \Rightarrow$ tank never overflows. $b_1 - b_2 < 0 \ \Rightarrow \dfrac{b_2}{b_1 - b_2} < 0$.

In (3) the expression, $\left(\dfrac{b_2}{b_1 - b_2} + 1 \right)^{-1}$ is feasible provided $\dfrac{b_2}{b_1 - b_2} \ \neq \ -1.$

Now the equation $\dfrac{b_2}{b1} - b_2 = \ -1 \Rightarrow \quad b_2 \ = - b_1 + b_2, \Rightarrow \ b_1 \ = \ 0 \Rightarrow$ no brine solution runs into the tank at any minute $\Rightarrow$ mixing or dilution does not take place; so we must have $\dfrac{b_2}{b_1 - b_2} \ \neq \ -1$, implying that (3) is feasible, provided $b_1 \neq b_2$ and the expression for $S(t)$ is preserved if $b_1 < b_2$.

(c) The tank will be empty at an instant t if $b_1 < b_2$ and $W_0 + (b_1 - b_2)t \ = \ 0,$ yielding $t \ = \ \dfrac{W_0}{b_2 - b_1}$.

However, this would render $S(t)$ undefined; thus the condition. $W_0 + (b_1 - b_2)t = 0$ is infeasible. This agrees with our intuition and the physics of the problem: as long as $b_1 > 0$ the tank is never empty.

The case $b_1 = b_2$ implies that the tank never overflows. $b_1 = b_2 \Rightarrow b_1 - b_2 = 0$. (2) yields:

$$\frac{dS}{dt} + \frac{b_2\ S}{W_0} = b_1 s_0$$

$$\Rightarrow I(t) = e^{\frac{b_2}{W_0}t}.$$

$$S(t) = e^{-\frac{b_2}{W_0}t}\left[\int b_1 s_0\, e^{\frac{b_2}{W_0}t}\ dt + C\right] = e^{\frac{-b_2}{W_0}t}\left[b_1 s_0 \frac{W_0}{b_2}\, e^{\frac{b_2}{W_0}t} + C\right]$$

$$S(0) = S_0 \Rightarrow \frac{b_1 s_0}{b_2}\ W_0 + C = S_0 \Rightarrow C = S_0 - \frac{b_1}{b_2} s_0 W_0$$

$$\Rightarrow S(t) = e^{\frac{-b_2}{W_0}t}\left[\frac{b_1 s_0}{b_2} W_0\, e^{\frac{b_2}{W_0}t} + S_0 - \frac{b_1 s_0}{b_2}\ W_0\right]$$

$$\Rightarrow S(t) = e^{\frac{-b_2}{W_0}t}\left[S_0 + \frac{b_1}{b_2} S_0 W_0\left(e^{\frac{b_2}{W_0}t} - 1\right)\right] \qquad (4)$$

IV. REFINEMENT OF THE MODEL TO A DELAY MODEL

Practical reality dictates that mixing cannot occur instantaneously throughout the tank. Thus the concentration of the brine leaving the tank at time t will be equal to the average concentration at some earlier instant, $t - h$ say, where $h > 0$. Setting $S(t) \equiv x(t)$ the ordinary differential equation (2) modifies to:

$$\dot{x}(t) = \frac{-b_2}{W_0 + (b_1 - b_2)t}\, x(t-h) + b_1 s_0 \qquad (5)$$

This is a delay differential type of linear nonhomogeneous type, in the form (1) with

$n = 1,\ A(t) \equiv 0,\quad B(t) = b(t) = \dfrac{-b_2}{W_0 + (b_1 - b_2)t}$, and $g(t) \equiv b_1 s_0$. Clearly *A*, *B* and *g* are continuous. For simplicity assume the following:

i. $s_0 = 0$. This implies that the inflow is fresh water

ii. $b_1 = b_2$. This implies that the inflow rate of fresh water equals the outflow rate of brine.

Then, set $C = \dfrac{b_2}{W_0}$ to obtain the autonomous homogeneous linear delay differential equation:

$$\dot{x}(t) = -c\, x(t-h). \qquad (6)$$

Above equation can be solved using the steps method. This consists in specifying appropriate initial conditions on prior intervals of length *h* and extending the solutions to the next intervals of length *h*. Since the tank contained S_0 lbs of salt thoroughly mixed in W_0 lbs of brine prior to time $t = t_0$, the commencement of the flow process, we can specify the initial conditions $x(t) = S_0$ for $t_0 - h \le t \le t_0$ and then obtain the solution on the intervals $\left[t_0 + (k-1)h, t_0 + kh\right]$ for $k = 1, 2, \cdots$ successively.

Therefore given:

$\dot{x}(t) = -c\,x(t-h)$ (7) and $x(t) = S_0$ on $[t_0 - h, t_0]$, we wish to obtain the solution for $t \geq t_0$.

Set $S_0 = \theta_0$. Note that $t_0 - h \leq t \leq t_0$ on $[t_0, t_0 + h]$. Therefore, $\dot{x}(t) - c\theta_0$ on $(t_0, t_0 + h)$, with initial condition:

$$x(t_0) = \theta_0 \Rightarrow x(t) = -c\theta_0 t + k_1;\ x(t_0) = \theta_0 = -c\theta_0 t_0 + k_1 \Rightarrow k_1 = (1 + c\,t_0)\theta_0$$

$$\Rightarrow x(t) = \theta_0[1 - c(t - t_0)] \quad \text{for } t \in [t_0, t_0 + h] \tag{8}$$

$$x(t) \geq 0,\ \text{if } 1 - ch \geq 0 \text{ or } ch \leq 1$$

On $[t_0 + h, t_0 + 2h]$, $t - h \in [t_0,\ t_0 + h]$. Hence,

$$\dot{x}(t) = -c\left[\theta_0 - c\theta_0(t - t_0 - h)\right] \text{ on } (t_0 + h, t_0 + 2h),$$ leading to the solution:

$$x(t) = -c\theta_0 t + \frac{c^2\theta_0}{2}\left[t - (t_0 + h)\right]^2 + k_2,\ \text{on } [t_0 + h,\ t_0 + 2h]. \tag{9}$$

By direct substitution and use of (8) we get:

$$x(t_0 + h) = -c\theta_0(t_0 + h) + k_2 = \theta_0\ (1 - ch)$$

$$\Rightarrow k_2 = \theta_0(1 + ct_0) \Rightarrow x(t) = \left(1 - c(t - t_0) + \frac{c^2}{2!}\left[t - (t_0 + h)\right]^2\right)\theta_0 \tag{10}$$

$$x(t) \geq 0,\ \text{if } 1 - c(t - t_0) \geq 0 \tag{11}$$

Noting that $-(t - t_0) \geq -2h$ on $[t_0 + h,\ t_0 + 2h]$, we infer that

$1 - c(t - t_0) \geq 0$ if $1 - 2ch \geq 0$ or $2ch \leq 1$. Hence $x(t) \geq 0$, if $ch \leq \frac{1}{2}\left(\text{if } ch \leq \frac{1}{2!}\right)$

Next, consider the interval $[t_0 + 2h, t_0 + 3h]$. Then $t - h \in [t_0 + h, t_0 + 2h]$. Therefore:

$$\dot{x}(t) = -c\left[1 - c\big(t - (t_0 + h)\big) + \frac{c^2}{2!}\left[t - (t_0 + 2h)\right]^2\right]\theta_0, \tag{12}$$

on the open set $(t_0 + 2h,\ t_0 + 3h)$.

Integrating over the interval $[t_0 + 2h,\ t_0 + 3h]$ yields:

$$x(t) = -c\left[t - c\left[\frac{t - (t_0 + h}{2}\right]^2 + \frac{c^2}{3!}\left[t - (t_0 + 2h)\right]^3\right]\theta_0 + k_3 \tag{13}$$

Direct substitution into (13) and use of (10) yields:

$$x(t_0 + 2h) = \left(-c[t_0 + 2h] + \frac{c^2}{2}h^2\right)\theta_0 + k_3 = \left[1 - 2ch + \frac{c^2h^2}{2}\right]\theta_0 \Rightarrow k_3 = (1 + c\,t_0)\theta_0$$

$$\Rightarrow k_3 = (1 + c\,t_0)\theta_0$$

$$\Rightarrow x(t) = \left(1 - c(t - t_0) + \frac{c^2}{2!}\left[t - (t_0 + h)\right]^2 - \frac{c^3}{3!}\left[t - (t_0 + 2h)\right]^3\right)\theta_0 \tag{14}$$

Assertion 1:

$$x(t) \geq 0, \text{on } [t_0 + 2h,\ t_0 + 3h],\ \text{if } ch \leq \frac{1}{3!} \tag{15}$$

Proof

$$t - (t_0 + h) \in [h, 2h] \text{ on } [t_0 + 2h, t_0 + 3h]$$

$$t-(t_0+2h)\in[0,h] \text{ on } [t_0+2h, t_0+3h]$$
$$t-t_0\in[2h,3h] \text{ on } [t_0+2h, t_0+3h]$$

From these facts, we deduce the following:

$$-c(t-t_0)\ge -3ch;\ -c[t-(t_0+2h)]\ge -ch, \tag{16}$$

$$-\frac{c^3}{3!}[t-(t_0+2h)]^3\ge -\frac{(ch)^3}{3!}, \tag{17}$$

$$\frac{c^2}{2!}[t-(t+h)]^2 \ \ge\ \frac{c^2h^2}{2!}. \tag{18}$$

Plug in (16), (17) and (18) into (14) to deduce that:

$$x(t)\ge\left(1-3ch+\frac{c^2h^2}{2!}-\frac{c^2h^2}{3!}\right)\theta_0$$

$$\ge\left(1-\frac{1}{2}-\frac{1}{6^4}\right)\theta \tag{19}$$

≥ 0, if $ch<\dfrac{1}{3!}$ proving assertion 1.

Let $x(t)\equiv y_k(t)$ on $[t_0+(k-1)h,\ t_0+kh]$. Then,

Theorem 1

$$x(t)\equiv y_k(t):$$

$$=\left[1\ +\ \sum_{j=1}^{k}\frac{(-1)^j\left(c[t-(t_0+(j-1)h)]\right)^j}{j!}\right]\theta_0 \tag{20}$$

on the interval $J_k=[t_0+(k-1)h, t_0+kh]$, with initial function $x(t)=y_{k-1}(t)$, on the interval J_{k-1} for k = 1,2, …, where, $x(t)=y_0=\theta_0$, on I_0. Moreover, $y_k(t)\ge 0$, wherever $ch\ \le\ \dfrac{1}{k!}$. Hence, $x(t)\ge 0,$ for $t\ge t_0$.

Proof

Proof is by inductive reasoning on k. The result is definitely true for k = 1, 2 and 3, following the solutions obtained on J_k, for k =1, 2 and 3. Assume that (20) is valid for $1\le k\le m$ for some integer m > 3. Then, $t-h\in J_m$ for $t\in J_{m+1}$ and hence $x(t-h)\ge 0$ on I_{m+1} if $ch\ \le\ \dfrac{1}{(m+1)!}$.

Now, $\dot{x}(t)=-c\,x(t-h)$ on $(t_0+mh\quad t_0+(m+1)h)\Rightarrow \dot{x}(t)=-c\,y_m(t-h)$

$$-c\left[1+\sum_{j=1}^{m}\frac{(-1)^j\left(c[t-h-(t_0+(j-1)h]\right)^j}{j!}\right]\theta_0\ =-c\left[1+\sum_{j=1}^{m}\frac{(-1)^j\left(c[t-(t_0+jh)]^j\right)}{j!}\right]\theta_0$$

Thus:

$$x(t) = -\left[c\,t + \sum_{j=1}^{m} \frac{(-1)^j \left(c\left[t - (t_0 + jh)\right]\right)^{j+1}}{(j+!)} \right]\theta_0 + k_{m+1}, \text{ on } J_{m+1} \qquad (21)$$

Plug $x(t_0 + mh)$ into (20) with $k = m$ and into (21) and set the results equal to each other to obtain,

$$\left[1 + \sum_{j=1}^{m} \frac{(-1)^j \left(c\left[t_0 + mh - (t_0 + (j-1)h)\right]\right)^j}{j!} \right]\theta_0$$

$$= \left[-c(t_0 + mh) + \sum_{j=1}^{m} \frac{(-1)^{j+1} \left(c\left[t_0 + mh - (t_0 + jh)\right]\right)^{j+1}}{(j+1)!} \right]\theta_0 \ + K_{m+1}$$

The last summation notation can be rewritten as:

$$\sum_{j=2}^{m} (-1)^j \left(c\left[t_0 + mh - \left(t_0 - (j-1)h\right)\right]^j \right) \equiv \sum_{j=2}^{m} T_j \text{ , so that}$$

$$K_{m+1} = \left[1 + c(t_0 + mh) - cmh\right]\theta_0 + \sum_{j=2}^{m}\left(T_j - T_j\right)\theta_0 = \left(1 + c\,t_0\right)\theta_0$$

Thus:

$$K_j = (1 + ct_0)\theta_0,\ j = 1, 2, \cdots \qquad (22)$$

Now plug (22) into (21) to obtain

$$x(t) = \left[ct + \sum_{j=1}^{m} \frac{(-1)^{j+1} \left(c\left[t - (t_0 + jh)\right]\right)^{j+1}}{(j+1)!} \right]\theta_0 + (1 + ct_0)\theta_0$$

$$= \left[1 - c(t - t_0) + \sum_{j=1}^{m} \frac{(-1)^{j+1} \left(c\left[t - (t_0 + jh)\right]\right)^{j+1}}{(j+1)!} \right]\theta_0$$

$$= \left[1 - c(t - t_0) + \sum_{j=2}^{m+1} \frac{(-1)^j \left(c\left[t - (t_0 + (j-1)h)\right]\right)^j}{j!} \right]\theta_0$$

$$= \left[1 + \sum_{j=1}^{m+1} \frac{(-1)\, j \left(c\left[t - (t_0 + (j-1)h)\right]\right)^j}{j!} \right]\theta_0, \qquad (23)$$

since $\sum_{j=1}^{m+1}\frac{(-1)^j\left(c\left[t-\left(t_0+(j-1)h\right)\right]\right)^j}{j!}$ evaluated at $j = 1$ yields $-c\left(t-t_0\right)$

Therefore, (20) is valid for all $k = 1, 2,\ldots\ldots$, as set out to be proved.

Next, we need to prove that; $y_k\left(t\right)\geq 0$ whenever $ch\leq\frac{1}{k!}$. From (20)

$$x\left(t\right)\geq\left[1-\sum_{\substack{1\leq j\leq k\\ j\text{ odd}}}\frac{\left(c\left[t-\left(t_0+(j-1)h\right)\right]\right)^j}{j!}\right]\theta_0,\text{ on } I_k \tag{24}$$

Observe that on $I_k, t-(t_0+(j-1)h)\in\left[t_0+(k-1)h-\left(t_0+(j-1)h\right),\ t_0+kh-\left(t_0+(j-1)h\right)\right]$,
that is, $t-\left(t_0+(j-1)h\right)\in\left[(k-j)h,\ (k+1-j)h\right]$.

Therefore, $t-\left(t_0+(j-1)h\right)\leq(k+1-j)h,\ \Rightarrow -c\left[t-\left(t_0+(j-1)h\right)\right]\geq-(k+1-j)ch$

Clearly, $x\left(t\right)\geq 0$ if:

$$\left[1-\sum_{\substack{1\leq j\leq k\\ j\text{ odd}}}\frac{\left[(k+1-j)ch\right]^{j+1}}{(j+1)!}\right]\theta_0\geq 0 \tag{25}$$

(25) would hold if:

$$\sum_{\substack{1\leq j\leq k\\ j\text{ odd}}}\frac{\left[(k+1-j)ch\right]^{j+1}}{(j+1)!}\leq 1 \tag{26}$$

Suppose that $ch\leq\frac{1}{k!}$ on J_k, then (26) would be valid if:

$$\sum_{\substack{1\leq j\leq k\\ j\text{ odd}}}\frac{(k+1-j)}{(j+1)!}\ \frac{1}{k!}\leq 1 \tag{27}$$

(27) would in turn be valid if:

$$\sum_{\substack{1\leq j\leq k\\ j\text{ odd}}}\frac{k}{k!}\frac{1}{(j+1)!}\ \leq 1, \tag{28}$$

that is, if:

$$\frac{1}{(k-1)!}\sum_{\substack{1\leq j\leq k\\ j\text{ odd}}}\frac{1}{(j+1)!}\ \leq 1 \tag{29}$$.

However, $(j+1)!\geq j^2$, so that $\frac{1}{(j+1)!}\leq\frac{1}{j^2}$.

Hence (29) would be true if:

$$\frac{1}{(k-1)!}\sum_{\substack{1\le j\le k \\ j \text{ odd}}}\frac{1}{j^2}\le 1 \qquad (30)$$

(30) would be valid if:

$$\frac{1}{(k-1)!}\sum_{j=1}^{k-1}1\le 1 \qquad (31)$$

$$\text{i.e. if } \frac{k-1}{(k-1)!}\le 1$$

$$\text{i.e. if } \frac{1}{(k-2)!}\le 1 \qquad (32)$$

Obviously (32) is valid for $k\ge 2.$

Combine this with the fact that $x(t)\ge 0$ for $t\in I$, if $ch\le 1$ to deduce that $y_k(t)\ge 0$ on I_k whenever $ch \le \frac{1}{k!}, k=1, 2,....$

The validity of (32) implies that

$$\lim_{k\to\infty}\frac{1}{(k-2)!}\le 1,\ \lim_{k\to\infty} y_k(t)\ge 0 \ \text{ or } x(t)\ge 0 \ \ \forall\, t\ge t_0.\ \text{ Note: } \left(0=\frac{1}{\infty}<1\right).$$

This completes the proof of the theorem.

V. ASSOCIATED NONHOMOGENOUS MODELS

Suppose that the inflow is not fresh water, that is $s_0\ne 0$. Then the nonhomogeneous differential difference equation:

$$\dot{x}(t)=-c\,x(t-h)+b_1 s_0, \qquad (33)$$

with initial data:

$$x(t)=\theta_0,\ t_0-h\le t\le t_0 \qquad (34)$$

referred to as the initial function, can be solved by a transformation of variables. In the sequel, let $y(t)=x(t)-\frac{b_1 s_0}{c}$. Then, $y(t) = \theta_0 - \frac{b_1}{c}s_0$ on $[t_0-h \quad t_0]$, $\dot{x}(t) = y(t)$ and

$$x(t-h) = y(t-h) + \frac{b_1}{c}s_0$$

$$\Rightarrow\ \dot{y}(t) = -c\left[y(t-h)+\frac{b_1}{c}s_0\right]+b_1 s_0 \Rightarrow \dot{y}(t) = -c\,y(t-h) \ \text{ with } y(t) = \theta_0-\frac{b_1}{c}s_0$$

on $[t_0-h,\ \ t_0]$

5.1 Corollary 1

i. The transformation $\tilde{y}(t) = x(t)-\frac{b_1}{c}s_0$ converts the linear nonhomogeneous delay differential equation (33) with initial function specification (34) to the following linear homogeneous delay equation with corresponding initial function:

$$\dot{\tilde{y}}(t) = -c\,\tilde{y}(t-h) \tag{35}$$

$$\tilde{y}(t) = \theta_0 - \frac{b_1}{c} s_0 \text{ on } [t_0 - h \quad t_0] \tag{36}$$

ii. If $x(t)$ is the solution of (35) on J_k then:

$$x(t) \equiv y_k(t) = \left[1 + \sum_{j=1}^{k} \frac{(-1)^j \left(c\left[t - (t_0 + (j-1)h)\right]\right)^j}{j!}\right]\phi + \frac{b_1 s_0}{c} \tag{37}$$

on J_k, with initial function $x(t) = y_{k-1}(t)$ on J_{k-1}, $k = 1, 2, ...$

where $x(t) = \phi = \theta_0 - \frac{b_1}{c} s_0$, on J_0.

Moreover, $y_{k-1}(t) \ge 0$ whenever $ch \le \frac{1}{k!}$. Hence, $x(t) \ge 0$, for $t \ge t_0$.

iii. $c\theta_0 - b_1 s_0 \ge 0$.

More generally, given the constant initial function problem:

$$\dot{x}(t) = a\,x(t) + b\,x(t-h) + c \tag{38}$$

$$x(t) = \theta_0,\ t_0 - h \le t \le t_0, \tag{39}$$

where $a, b, c,$ are given constants, the change of variables $\tilde{y}(t) = x(t) + d$

$$\Rightarrow\ \dot{\tilde{y}}(t) = \dot{x}(t) \text{ and } \dot{\tilde{y}}(t) = a\left[\tilde{y}(t) - d\right] + b\left[\tilde{y}(t-h) - d\right] + c$$

$$\Rightarrow\ \dot{\tilde{y}}(t) = a\,\tilde{y}(t) + b\,\tilde{y}(t-h) + c - (a+b)d$$

Setting $c - (a+b)d = 0 \Rightarrow d = \frac{c}{a+b}$. Also $\tilde{y}(t) = \theta_0 + \frac{c}{a+b}$, leading to the following proposition.

5.2 Proposition 1

The initial function problem:

$$\dot{x}(t) = ax(t) + bx(t-h) + c \tag{40}$$

$$x(t) = \theta_0,\ t_0 - h \le t \le t_0 \tag{41}$$

where a, b and c are given constants such that $a + b \ne 0$, is equivalent to the initial function problem of the linear homogeneous type:

$$\dot{\tilde{y}}(t) = a\tilde{y}(t) + b\tilde{y}(t-h) \tag{42}$$

$$\tilde{y}(t) = \phi_0,\ t_0 - h \le t \le t_0 \tag{43}$$

where:

$$\phi_0 = \theta_0 + \frac{c}{a+b}. \tag{44}$$

Furthermore $x(t)$ is related to $\tilde{y}(t)$ by the equation:

$$x(t)=\tilde{y}(t)-\frac{c}{a+b} \qquad (45)$$

Remark 1
Instead of solving (40) and (41), solve the easier equivalent problem (42) through (45).
If $b = 0$, (40) and (41) degenerate to the IVP

$$\dot{x}=a\,x+c$$
$$x(t_0)=\theta_0,$$

with the unique solution $x(t)=\left(\theta_0+\frac{c}{a}\right)e^{a(t-t_0)}-\frac{c}{a}.$

Equivalently, the solution of (4) through (45) is given by:

$$x(t)=\tilde{y}(t)-\frac{c}{a}$$
$$\Rightarrow x(t)=\phi_o\, e^{a(t-t_0)}-\frac{c}{a}\ =\left(\theta_0+\frac{c}{a}\right)e^{a(t-t_0)}-\frac{c}{a}, \qquad (46)$$

as desired.
Remark 2
The transformation (45) is doomed if c in (40) is replaced by nonconstant $c(t)$, as $\tilde{y}(t) = x(t) + d(t)$ leads to $c(t) - ad(t) - bd(t-h) = 0$, and it is impossible to determine $d(t)$.

Linear translation of the initial interval and representation of the unique solution.
Consider the constant initial function problem:

$$\dot{x}(t)=ax(t)+bx(t-h),\ t\ge t_0, \qquad (47)$$
$$x(t)=\theta_0,\ t_0-h\le t\le t_0 \qquad (48).$$

Define:

$$z(t)=x(t-t_0). \qquad (49)$$

Then $z(t_0)=x(t_0-t_0)=x(0)$ and $z(t_0-h)=x(t_0-h-t_0)=x(-h)$.

Also: $t\in[t_0-h,t_0]\Rightarrow\ t-t_0\in[-h,0]$ and

$$x(t)=z(t_0+t)$$

$$\dot{z}(t)=\dot{x}(t-t_0)$$
$$\Rightarrow \dot{z}(t)=a\,z(t)+b\,z(t-h),\,t\ge 0 \qquad (50)$$
$$z(t)=\theta_0,\,-h\le t\le 0 \qquad (51)$$

Therefore, the constant initial function problem (47), (48) is equivalent to the constant initial function problem (50), (51) where $x(t)$ is related to $z(t)$ through the equation:

$$x(t) = z(t+t_0). \qquad (52)$$

Consequently, without any loss in generality, given (47), (48) we can solve the equivalent problem:

$$\dot{x}(t) = a\,x(t) + b\,x(t-h),\, t \geq 0 \qquad (53)$$

$$x(t) = \theta_0, -h \leq t \leq t_0 \qquad (54)$$ Denote:

$$x(\cdot) \to x(\cdot + h) \qquad (55)$$

$$J_k = \left[(k-1)h,\ kh\right],\ k = 0,1,2, \qquad (56)$$

$$J_k^0 = \left((k-1)h,\ kh\right),\ k = 1,2,.. \qquad (57)$$

Then, $t \in J_k^0 \Rightarrow t-h \in J_{k-1}^0$; consequently $\dot{x}(t) = a\,x(t) + b\theta_0$, on J_1^0.

Using (46) with *c* replaced by $b\theta_0$ we obtain $x(t) = \left(\theta_0 + \frac{b\,\theta_0}{a}\right)e^{a\,t} - \frac{b\,\theta_0}{a}$ on J_1.

Thus:

$$x(t) = \left[-\frac{b}{a} + \left(1+\frac{b}{a}\right)e^{at}\right]\theta_0 = \left[c_1 + c_{11}\,e^{a\,t}\right]\theta_0, \text{ on } J_1 \qquad (58)$$

Clearly, $x(0) = \theta_0$ and $x(h) = \left(\theta_0 + \frac{b}{a}\theta_0\right)e^{a\,h} - \frac{b}{a}\theta_0.$

Consider $t \in J_2$. Then on J_2^0,

$$\dot{x}(t) = a\ x(t) + \left[-\frac{a}{b} + \left(1+\frac{b}{a}\right)\ e^{a(t-h)}\right]\theta_0$$

Denote the integrating factor by $I(t)$. Then, $I(t) = e^{-a\,t}$

Hence:

$$x(t) = e^{a\,t}\left(\left[\int -\frac{b}{a}\,e^{-a\,t}dt + \int\left(1+\frac{b}{a}\right)e^{-a\,h}dt\right]\theta_0 + C\right)$$

$$= e^{a\,t}\left(\left[\frac{b}{a^2}\,e^{-a\,t} + \left(1+\frac{b}{a}\right)e^{-a\,h}\,t\right]\theta_0 + C\right) \qquad (59)$$

$$x(h) = \left[\left(1+\frac{b}{a}\right)e^{a\,h} - \frac{b}{a}\right]\theta_0 \qquad \text{(using (58))}$$

$$= e^{a\,h}\left(\left[\left(1+\frac{b}{a}\right)h\,e^{-a\,h} + \frac{b}{a^2}\,e^{-a\,h}\right]\theta_0 + C\right) \quad \text{(using (59))}$$

$$\Rightarrow C=\left[-\frac{b}{a}e^{-ah}+\left(1+\frac{b}{a}\right)\left(1-h\,e^{-ah}\right)-\frac{b}{a^2}\,e^{-ah}\right]\theta_0$$

$$\Rightarrow x(t)=e^{at}\left[\frac{b}{a^2}\,e^{-at}+\left(1+\frac{b}{a}\right)e^{-ah}t-\frac{b}{a}\,e^{-ah}-\frac{b}{a^2}\,e^{-ah}+\left(1+\frac{b}{a}\right)\left(1-he^{-ah}\right)\right]\theta_0$$

$$=e^{at}\left[\left[\left(1+\frac{b}{a}\right)t-\frac{b}{a}-\frac{b}{a^2}-h\left(1+\frac{b}{a}\right)\right]e^{-ah}+1+\frac{b}{a}+\frac{b}{a^2}e^{-at}\right]\theta_0$$

$$=\frac{b}{a^2}+\left[1+\frac{b}{a}-\left(\frac{b}{a}+\frac{b}{a^2}+h\left(1+\frac{b}{a}\right)\right)e^{-at}+\left(1+\frac{b}{a}\right)e^{-ah}t\right]e^{at}\,\theta_0$$

$$=\left[c_2+\left(c_{21}+c_{22}t\right)e^{at}\right]\theta_0 \quad \text{on } J_2 \tag{60}$$

where:

$$c_2=\frac{b}{a^2},\ c_{21}=1+\frac{b}{a}-\left(\frac{b}{a}+\frac{b}{a^2}+h\left(1+\frac{b}{a}\right)\right)e^{-ah},\ c_{22}=\left(1+\frac{b}{a}\right)e^{-ah} \tag{61}$$

Notice that:

$$c_2=-\frac{c_1}{a}=\left(-\frac{1}{a}\right)c_1,\ c_{21}=\left[\left(1+\frac{1}{a}\right)e^{-ah}\right]c_1+\left[1+he^{ah}\right]c_{11},\ c_{22}=e^{-ah}\,c_{11}. \tag{62}$$

We propose the following result.

5.3 Theorem 2:

$$x(t)=\left[c_k+\left(\sum_{j=1}^{k}c_{kj}\,t^{j-1}\right)e^{at}\right]\theta_0 \tag{63}$$

on J_k for appropriately determinable constants $c_k,\,c_{kj},\ j=1,2,...,k,$ where:

$$c_1=-\frac{b}{a},\ c_{11}=1+\frac{b}{a};\ c_k=(-1)^k\,\frac{b^{k-1}}{a^k},\ c_{kk}=\frac{1}{(k-1)!}b^{k-2}e^{-(k-1)ah}c_{11};k=2,3,\cdots, \tag{64}$$

and for $j\neq k, c_{kj}$ depends on $h,$ but has no general mathematical representation.

Proof

The proof is by inductive reasoning. From (58), we see that the theorem is valid on J_1, with $c_1=-\frac{b}{a},\ c_{11}=1+\frac{b}{a}.$ From (60), (61) and (62), it is clear that the theorem is true on J_2, with:

$$c_2=\frac{b}{a^2},c_{21}=1+\frac{b}{a}-\left(\frac{b}{a}+\frac{b}{a^2}+h\left(1+\frac{b}{a}\right)\right)e^{-ah},c_{22}=\left(1+\frac{b}{a}\right)e^{-ah}=\frac{1}{(2-1)!}b^{2-2}e^{-(2-1)ah}\quad c_{11} \tag{65}$$

Consider J_3. On $t\in J_3^0\Rightarrow t-h\in J_2^0.$ By (63), we have:

$$x(t-h)=\left[c_2+\left(c_{21}+c_{22}(t-h)\right)e^{a(t-h)}\right]\theta_0\Rightarrow \dot{x}(t)=ax(t)+b\left[c_2+\left(c_{21}+c_{22}(t-h)\right)e^{a(t-h)}\right]\theta_0$$

Integrating factor, $I(t) = e^{-at}$

$$\Rightarrow x(t) = e^{at}\left[\int b\,c_2\,e^{-at}dt + \int b\,c_{21}e^{a(t-h)}\,e^{-at}d\,t + \int b\,c_{22}(t-h)e^{a(t-h)}e^{-at}dt + C\right]\theta_0$$

$$= e^{at}\left[-\frac{b\,c_2}{a}\,e^{-at} + b\,c_{21}\,e^{-ah}t + b\,c_{22}\left(\frac{t^2}{2} - ht\right)e^{-ah} + C\right]\theta_0 \qquad (66)$$

Now plug $x(2h)$ into (63) and (66) to obtain *C* as follows:

$$x(2h) = e^{2ah}\left[-\frac{bc_2}{a}\,e^{-2ah} + bc_{21}\,e^{-ah}\,2h + bc_{22}\left(2h^2 - 2h^2\right)e^{-ah} + C\right]\theta_0$$

$$= \left[-\frac{bc_2}{a} + bc_{21}\,h\,e^{ah} + C\,e^{2ah}\right]\theta_0$$

$$= \left[c_2 + \left(c_{21} + 2h\,c_{22}\right)e^{2ah}\right]\theta_0$$

$$\Rightarrow C = e^{-2ah}\left[\frac{bc_2}{a} - bc_{21}\,h\,e^{ah} + c_2 + \left(c_{21} + 2h\,c_{22}\right)e^{2ah}\right]$$

Now, plug this value of C into (66) and set the resulting expression equal to $\left[c_3 + \left(c_{31} + c_{32}\,t + c_{33}\,t^2\right)e^{at}\right]\theta_0$ to get:

$$x(t) = e^{at}\begin{bmatrix} -\dfrac{b\,c_2}{a}\,e^{-at} + b\,c_{21}e^{-ah}t + b\,c_{22}\left(\dfrac{t^2}{2} - ht\right)e^{-ah} \\ +\dfrac{b\,c_2}{a}\,e^{-2ah} - b\,c_{21}\,h\,e^{ah} + c_2\,e^{-2ah} + c_{21} + 2hc_{22} \end{bmatrix}\theta_0$$

$$= \left[c_3 + \left(c_{31} + c_{32}t + c_{33}\,t^2\right)\right]\theta_0 \qquad (67)$$

$$\Rightarrow c_3 = -\frac{b\,c_2}{a} = -\frac{b}{a}\left(\frac{b}{a^2}\right) = -\frac{b^2}{a^3}$$

$$= -\frac{b}{a}\left(-\frac{c_1}{a}\right) = c_1\,c_2 = c_1\left(-\frac{c_1}{a}\right) = -\frac{{c_1}^2}{a}$$

Clearly (67) $\Rightarrow$

$$c_3 = (-1)^3\,\frac{b^{3-1}}{a^3},\; c_{33} = \frac{1}{2}b\,c_{22}\,e^{-ah} = \frac{1}{2}\,b\,e^{-2ah}\,c_{11} = \frac{1}{(3-1)!}\,b^{3-2}\,e^{-(3-1)ah}$$

$$c_{32} = b\,c_{21}e^{-ah} - b\,h\,c_{22}e^{-ah} = \left(1 + \frac{1}{a}\right)be^{-2ah}c_1 + b\left(1 + h\,e^{-ah}\right)e^{-ah}c_{11} - b\,he^{-2ah}\,c_{11}$$

$$= b\left[\left(1 + \frac{1}{a}\right)c_1 + e^{-ah}\,c_{11}\right]e^{-2ah} \qquad (68)$$

$$c_{31} = \frac{b\,c_2}{a}\,e^{-2ah} - b\,c_{21}\,he^{ah} + c_2e^{-2ah} + c_{21} + 2\,h\,c_{22}$$

$$=\left(-\frac{b}{a}\right)\frac{c_1}{a}e^{-2ah}+\left(1-b\,h\,e^{ah}\right)\left[\left(1+\frac{1}{a}\right)e^{-ah}\,c_1+\left(1+h\,e^{-ah}\right)c_{11}\right]-\frac{c_1}{a}e^{-2ah}+2h\left(1+\frac{b}{a}\right)e^{-ah}$$
$$=\frac{1}{a}e^{-2ah}\,c_1^2+2\,hc_{11}\,e^{-ah}-\frac{1}{a}e^{-2ah}c_1+\left(1-bhe^{ah}\right)\left[\left(1+\tfrac{1}{a}\right)e^{-ah}c_1+\left(1+h\,e^{-ah}\right)c_{11}\right]$$
$$=\frac{1}{a}e^{-2ah}\,c_1^2+\left[\left(1+\frac{1}{a}\right)e^{-ah}-b\,h\left(1+\frac{1}{a}\right)-\frac{1}{a}e^{-2ah}\right]c_1$$
$$+\left[2\,h\,e^{-ah}+\left(1+h\,e^{-ah}\right)-b\,h\,e^{ah}-b\,h^2\right]c_{11} \qquad (69)$$

Therefore the theorem is also valid on J_3. From the results already obtained for c_{21}, c_{31} and c_{32}, it is clear that no definite pattern can be postulated for $c_{kj},\ j\in\{1,2,....,k-1\}$, even for the simplest initial function problem.

Now, we proceed to complete the proof of theorem.

Assume that the theorem is valid on $J_k, k\ge 4$. $t\in J_{k+1}^0 \Rightarrow t-h\in J_k^0 \Rightarrow$ the theorem is valid with t replaced by $t-h$ for $t\in J_{k+1}^0$. Hence:

$$\dot{x}(t)=a\,x(t)+b\left[c_k+\left(\sum_{j=1}^{k}c_{kj}\,(t-h)^{j-1}\right)e^{a(t-h)}\right]\theta_0 \qquad (70)$$

$$I(t)=e^{-at}$$

$$\Rightarrow x(t)=e^{at}\left[\int b\,c_k\,e^{-at}\,dt+\int\left(\sum_{j=1}^{k}c_{kj}\,(t-h)^{j-1}\,e^{-ah}\,dt+C\right)\right]\theta_0$$
$$=e^{at}\left[-\frac{1}{a}b\,c_k\,e^{-at}+\left(\sum_{j=1}^{k}c_{kj}\,\frac{(t-h)^j}{j}e^{-ah}+C\right)\right]\theta_0 \qquad (71)$$

The constant term is:

$$-\frac{b}{a}c_k\,\theta=\left(-\frac{b}{a}\right)(-1)^k\frac{b^{k-1}}{a^k}\theta_0\quad\text{using }((71))$$
$$=(-1)^{k+1}\frac{b^k}{a^{k+1}}\theta_0=(-1)^{k+1}\frac{b^{(k+1)-1}}{a^{k+1}}\theta_0=c_{k+1}\theta_0\Rightarrow c_{k+1}=(-1)^{k+1}\frac{b^{(k+1)-1}}{a^{k+1}} \qquad (72)$$

The term in $e^{at}t^k$ is $e^{at}\,e^{-ah}\,\dfrac{t^k}{k}\,c_{kk}\theta=e^{at}\,\dfrac{t^k}{k}\theta_0\,e^{-ah}\,\dfrac{b}{(k-1)!}\,b^{k-2}\,e^{-(k-1)ah}c_{11}$,

by the induction hypothesis. Set

$$d_{k+1,k+1}=e^{at}\,\frac{1}{k}\,e^{-ah}\,\frac{b}{(k-1)!}\,b^{k-2}\,e^{-(k-1)ah}c_{11}=e^{at}\,\frac{1}{k}\,e^{-ah}\,\frac{b}{(k-1)!}\,b^{k-2}\,e^{-(k-1)ah}c_{11}.\text{ Then}$$

$$d_{k+1,k+1}=\frac{b^{k-1}}{k!}\,e^{-kah}\,c_{11}=\frac{b^{(k+1)-2}}{(k+1-1)!}\,e^{-(k+1-1)ah}\,c_{11}=c_{k+1,k+1}\text{, as desired.}$$

Note that $(t-h)^j=\sum_{r=0}^{j}\binom{j}{r}t^{j-r}(-h)^r$.

VI. CONCLUSION

This article gave an exposition on how a delay could be incorporated into an ordinary differential equations dilution model to yield a delay differential equations dilution model. It went on to formulate and prove appropriate theorems on solutions and feasibility of such models. It also showed how a nonhomogeneous model with constant initial function could be converted to a homogeneous model. Some of the results relied on the use of integrating factors, change of variables technique and the principle of mathematical induction.

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