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Duality between initial and terminal function problems

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ABSTRACT: *This paper successfully investigated the topological duality between initial and terminal function problems with respect to single-delay autonomous linear control systems using the expressions for the solution matrices and indices of control systems developed and proved in Ukwu and Garba [2014a, c] for single – delay autonomous linear systems, as well as the general variation of constants formula for the relevant systems. The paper obtained independent proofs that the solution matrices and the indices of control systems are analytic almost everywhere and differentiable except at a single point. The proofs were achieved using an algorithm involving key transformations from one problem to the other, effectively establishing that the results from one problem can be realized from the other.*

KEYWORDS: *Duality, Initial, Problems, Terminal, Transformations.*

I. INTRODUCTION

The qualitative approach to the controllability of functional differential control systems have been areas of active research for the past fifty years among control theorists and applied mathematicians in general. This circumvents the severe difficulties associated with the search for and computations of solutions of such systems.

Unfortunately computations of solutions cannot be wished away in the tracking of trajectories and many practical applications. Literature on state space approach to control studies is replete with variation of constants formulas, which incorporate the solution matrices of the free part of the systems. See Chukwu (1992), Gabsov and Kirillova (1976), Hale (1977), Manitius (1978), Tadmore (1984), Ukwu (1987, 1992, 1996) and Ukwu and Garba [2014a]. Furthermore the importance of indices of control systems matrices stems from the fact that they not only pave the way for the derivation of determining matrices for the determination of Euclidean controllability and compactness of cores of Euclidean targets but can be used independently for such determination, Ukwu and Garba [2014c]. Regrettably no author had made any noteworthy attempt to obtain general expressions for such solution matrices and indices of control systems matrices involving the delay h until Ukwu and Garba [2014a,c].

This article makes further contribution to knowledge in controllability studies by establishing that initial and terminal function problems are duals of each other, using Ukwu and Garba [2014a,c] as key components in the ensuing analyses and much more.

II. PRELIMINARIES

Consider the initial function problem:

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) \quad (1)$$

$$x(t) = \phi(t),\ t \in [-h, 0] \quad (2)$$

and the terminal function problem:

$$\dot{y}(\tau) = -y(\tau)A_0 - y(\tau+h)A_1 \quad (3)$$

$$y(\tau) = \psi(\tau),\ \tau \in [t_1 - h, t_1] \quad (4)$$

where A_0 and A_1 are $n \times n$ constant matrices; $x(.)$ and $y(.)$ are column and row n-vector functions respect ively.

Let $Y(t,\tau)$ and $X(\tau,t)$ be solution matrices of systems (1) and (3) respectively. Then $X(\tau,t)$ and $Y(t,\tau)$ satisfy respectively the following matrix differential equations:

$$\frac{\partial}{\partial t}Y(t,\tau)=A_0Y(t,\tau)+A_1Y(t,\tau+h),\text{ where } Y(t,\tau)=\begin{cases}I_n,\ t=\tau\\0,\ t<\tau\end{cases}\qquad(5)$$

$$\frac{\partial}{\partial \tau}X(\tau,t)=-X(\tau,t)A_0-X(\tau+h,t)A_1,\qquad(6)$$

for $0<\tau<t,\ \tau\neq t-kh,\ k=0,1,\ldots$ where

$$X(\tau,t)=\begin{cases}I_n;\ \tau=t\\0;\ \tau>t\end{cases}\qquad(7)$$

See Chukwu (1992), Hale (1977) and Tadmore (1984) for properties of $X(t,\tau)$. of particular importance is the fact that $\tau\to X(\tau,t)$ is analytic on the intervals $(t_1-(j+1)h,\ t_1-jh),\ j=0,1,\ldots;\ t_1-(j+1)h>0$. Any such $\tau\in(t_1-(j+1)h,\ t_1-jh)$ is called a regular point of $\tau\to X(t,\tau)$.

The first result we will establish is that $Y(t,\tau)$ and $X(\tau,t)$ are topological duals of each other. This result will rely on and exploit the following Ukwu-Garba's theorems on global expressions for the matrices $Y(t,\tau)$ and $X(\tau,t)$. See Ukwu and Garba [2014 a,c].

Let $J_{k-i}=[(k-i)h,(k+1-i)h],k\in\{0,1,\cdots\},i\in\{0,1\}$; let $K_j=[t_1-(j+1)h,\ t_1-jh]$,
$j\geq 0: t_1-(j+1)h\geq 0.$

III. THEOREM: UKWU-GARBA'S SOLUTION MATRIX FORMULA FOR AUTONOMOUS, SINGLE – DELAY LINEAR SYSTEMS

$$Y(t)=\begin{cases}e^{A_0t},\ t\in J_0; & (8)\\ e^{A_0t}+\int_h^t e^{A_0(t-s)}A_1e^{A_0(s-h)}ds,\ t\in J_1; & (9)\\ e^{A_0t}+\int_h^t e^{A_0(t-s)}A_1e^{A_0(s-h)}ds & \\ +\left[\sum_{j=2}^{k}\int_{jh}^{t}e^{A_0(t-s_j)}\prod_{i\in\{j,j-1,\cdots,2\}}\int_{(i-1)h}^{s_i-h}A_1e^{A_0(s_i-h-s_{i-1})}\right]A_1e^{A_0(s_1-h)}\prod_{\lambda=1}^{j}ds_\lambda,t\in J_k,k\geq 2 & (10)\end{cases}$$

IV. THEOREM: UKWU-GARBA'S CONTROL INDEX THEOREM FOR AUTONOMOUS LINEAR DELAY CONTROL SYSTEMS:

$$X(\tau,t_1)=\begin{cases}e^{A_0(t_1-\tau)},\ \tau\in K_0; & (11)\\ e^{A_0(t_1-\tau)}-\int_{t_1-h}^{\tau}e^{A_0(t_1-h-s_1)}A_1e^{A_0(s_1-\tau)}ds_1,\ \tau\in K_1; & (12)\\ e^{A_0(t_1-\tau)}-\int_{t_1-h}^{\tau}e^{A_0(t_1-h-s_1)}A_1e^{A_0(s_1-\tau)}ds_1 & (13)\\ +\left[\sum_{k=2}^{j}(-1)^j\int_{t_1-kh}^{\tau}e^{A_0(t_1-h-s_1)}\prod_{i\in\{k,k-1,\cdots,2\}}\int_{(i-1)h}^{s_i-h}\left[\prod_{q=1}^{k-1}A_1e^{A_0(s_q-h-s_{q+1})}\right]A_1e^{A_0(s_k-\tau)}\right]\prod_{\lambda=1}^{k}ds_\lambda,\tau\in K_j,j\geq 2 & \end{cases}$$

Our current result is captured in the following theorem.

V. THEOREM: UKWU'S TOPOLOGICAL DUAL ALGORITHM FOR AUTONOMOUS LINEAR DELAY SYSTEMS

$Y(t,\tau)$ and $X(\tau,t)$ are topological duals of each other subject to the transformations below. To obtain $X(\tau,t)$ from $Y(t,\tau)$) perform the following operations;

(i) $t \to \; t-\tau$ in the formula for $Y(t)$.

(ii) $(.) \to t-(.)$ in the lower integral limits, as well as in the leading integral limits in (10) with the preceding sign convention $(-1)^j$, for $1 \le j \le k$, followed by the notational changes

$$j \to k,\ J_k + \tau \to K_j;\quad \sum_{j=2}^{k}(.) \to \sum_{k=2}^{j}(.)$$

(iii) $s_i - h \to s_i + h$ in the upper integral limits in $Y(t,\tau)$.

(iv) The leading exponential function $e^{A_0(t-\tau-s_j)} \to e^{A_0(t-h-s_1)}, \forall j \in \{2,3,\cdots k\}$, in the integrals in $Y(t,\tau)$ and the trailing exponential function $e^{A_0(s_1-h)} \to e^{A_0(s_j-\tau)}$.

To obtain $Y(t,\tau)$ from $X(\tau,t)$ perform the following operations:

(v) $(.) \to t-(.)$ in the leading integral limits, as well as in the lower integral limits. Omit the sign convention $(-1)^j$, for $1 \le j \le k$, and effect the notational changes $K_j \to J_k + \tau;\quad \sum_{k=2}^{j}(.) \to \sum_{j=2}^{k}(.)$

(vi) Replace the leading exponential function $e^{A_0(t-h-s_1)}$ by $e^{A_0(t-\tau-s_j)}, \forall j \in \{1,2,\cdots k\}$, in $X(\tau,t)$ and the trailing exponential function $e^{A_0(s_j-\tau)}$ by $e^{A_0(s_1-h)}$

(vii) Replace the upper integral limits $s_i + h$ in $X(\tau,t)$ by $s_i - h$.

(viii) Given the initial function problem (1) and (2) the equivalent terminal function problem is given by:

$$\dot{y}(\tau) = -y(\tau)A_0 - y(\tau+h)A_1 \qquad (14)$$

$$y(\tau) = \psi(\tau),\ \tau \in [t_1-h, t_1] \qquad (15)$$

where

$$\psi(\tau) = \phi^{\mathrm{T}}(\tau - t_1), \tau \in [t_1-h, t_1]; (.)^{\mathrm{T}} \text{ denotes the transpose of } (.). \qquad (16)$$

Proof

We need to prove that the continuity and analytic dispositions of $X(\tau,t_1), \tau \in K_j$ can be inferred from those of $Y(t,\tau), t \in J_k + \tau$ and vice-versa. In particular, we prove that the continuity and non-analytic behavior of $X(\tau,t_1)$, for $\tau = t_1 - jh : t_1 - (j+1)h > 0$ correspond to those of $Y(t,\tau)$, for $t = \tau + kh : t < t_1$.

We start from the expression for $Y(t,\tau)$. Note that $Y(t,\tau) = Y(t-\tau)$ since the systems (1) and (5) are autonomous. So we may replace t by $t-\tau$ in Ukwu and Garba's solution matrix theorem to secure $Y(t,\tau)$. This justifies (i). With $t = t_1$, in Ukwu and Garba's Control Index theorem, the transformation $(.) \to t-(.)$, in (ii) ensures that $t \to t_1 - \tau,\ jh \to t_1 - jh,\ t-\tau \to \tau, (i-1)h \to t_1 - (i-1)h$.

These, together with the sign convention and the notational changes in (ii) guarantee that (8) and (9) with t replaced by $t-\tau$ transform to (11) and (12) respectively. We need to prove that the multiple integral in (10) transforms to the multiple integral in (13). To achieve the proof, combine the foregoing with $j \to k$ and the transformations in (iii) and (iv) to deduce immediately that the multiple integral in (10) transforms to the multiple integral in (13), proving that $X(\tau,t_1)$ has been obtained from $Y(t,\tau)$. The proof that $X(\tau,t_1)$ transforms to $Y(t,\tau)$ is similar.

Finally we investigate the continuity and analytic dispositions of $Y(t,\tau)$ and $X(\tau,t_1)$. Being sums of integrals of products of constant matrices and exponential functions we deduce that
$Y(t,\tau)$ and $X(\tau,t_1)$ are analytic in the interiors of $J_k+\tau$ and K_j respectively, since $e^{A_0(.)}$ is C^∞.
Finally we must examine $Y(t,\tau)$ and $X(\tau,t_1)$ on the boundaries of $J_k+\tau$ and K_j respectively.

$$Y(t,\tau)=0\ \forall t<\tau \Rightarrow \lim_{t\to\tau^-} Y(t,\tau)=0, \lim_{t\to\tau^-}\dot{Y}(t,\tau)=0;$$

$$Y(t,\tau)=e^{A_0 t}\ \forall t\in[\tau,\tau+h] \Rightarrow \lim_{t\to\tau^+} Y(t,\tau)=I,\ \lim_{t\to\tau^+}\dot{Y}(t,\tau)=A_0.$$

Therefore $Y(t)$ is not continuous at $t=\tau$ and certainly not differentiable there. It follows that $Y(t)$ is not analytic at $t=\tau$.

$$\lim_{t\to(\tau+h)^-}\dot{Y}(t,\tau)=A_0e^{A_0h};\ \lim_{t\to(\tau+h)^+}\dot{Y}(t,\tau)=A_0\lim_{t\to(\tau+h)^+}Y(t,\tau)+A_1\lim_{t\to(\tau+h)^+}Y(t,\tau+h)=A_0Y(h)+A_1Y(0)=A_0e^{A_0h}+A_1,$$

by the continuity of $Y(t)$ for $t>0$ and the appropriate evaluations using the interval $J_0+\tau$ for the left limit and $J_1+\tau$ for the right limit. Alternatively, we apply Leibniz's rule for differentiating an integral to $\lim_{t\to(\tau+h)^+}Y(t,\tau)$ to obtain $A_0e^{A_0(2h-h)}+A_1e^{A_0(h-h)}-0+\int_{\tau+h}^{\tau+h}A_0e^{A_0(t-s)}A_1e^{A_0(t-s)}ds=A_0e^{A_0h}+A_1.$

Therefore $Y(t,\tau)$ is not differentiable at $t=\tau+h$ and hence not analytic there. Equivalently using the fact that (1) and (5) are autonomous, $Y(t)$ is not differentiable at $t=h$ and hence not analytic there.

$$Y(t)=e^{A_0t}+\int_h^t e^{A_0(t-s)}A_1e^{A_0(s-h)}ds$$
$$+\left[\sum_{j=2}^{k}\int_{jh}^{t}e^{A_0(t-s_j)}\prod_{i\in\{j,j-1,\cdots,2\}}\int_{(i-1)h}^{s_i-h}A_1e^{A_0(s_i-h-s_{i-1})}\right]A_1e^{A_0(s_1-h)}\prod_{\lambda=1}^{j}ds_\lambda,\ t\in J_k,\ k\ge 2$$

$$\Rightarrow \lim_{t\to(kh)^+}Y(t)=e^{A_0kh}+\int_h^{kh}e^{A_0(kh-s)}A_1e^{A_0(s-h)}ds$$
$$+\left[\sum_{j=2}^{k}\int_{jh}^{kh}e^{A_0(kh-s_j)}\prod_{i\in\{j,j-1,\cdots,2\}}\int_{(i-1)h}^{s_i-h}A_1e^{A_0(s_i-h-s_{i-1})}\right]A_1e^{A_0(s_1-h)}\prod_{\lambda=1}^{j}ds_\lambda,\ k\ge 2$$

$$\lim_{t\to(kh)^+}Y(t)=e^{A_0kh}+\int_h^{kh}e^{A_0(kh-s)}A_1e^{A_0(s-h)}ds$$
$$+\left[\sum_{j=2}^{k-1}\int_{jh}^{kh}e^{A_0(kh-s_j)}\prod_{i\in\{j,j-1,\cdots,2\}}\int_{(i-1)h}^{s_i-h}A_1e^{A_0(s_i-h-s_{i-1})}\right]A_1e^{A_0(s_1-h)}\prod_{\lambda=1}^{j}ds_\lambda,\ k-1\ge 2$$

since the integral contribution at $j=kh$ is zero.

On J_{k-1}, $Y(t)=e^{A_0t}+\int_h^t e^{A_0(t-s)}A_1e^{A_0(s-h)}ds$

$$+\left[\sum_{j=2}^{k-1}\int_{jh}^{t}e^{A_0(t-s_j)}\prod_{i\in\{j,j-1,\cdots,2\}}\int_{(i-1)h}^{s_i-h}A_1e^{A_0(s_i-h-s_{i-1})}\right]A_1e^{A_0(s_1-h)}\prod_{\lambda=1}^{j}ds_\lambda,\ k-1\ge 2$$

$$\Rightarrow \lim_{t\to(kh)^-} Y(t) = e^{A_0 kh} + \int_h^{kh} e^{A_0(kh-s)} A_1 e^{A_0(s-h)} ds$$

$$+\left[\sum_{j=2}^{k-1}\int_{jh}^{kh} e^{A_0(kh-s_j)} \prod_{i\in\{j,j-1,\cdots,2\}} \int_{(i-1)h}^{s_i-h} A_1 e^{A_0(s_i-h-s_{i-1})}\right] A_1 e^{A_0(s_1-h)} \prod_{\lambda=1}^{j} ds_\lambda,\ k-1\geq 2$$

This completes the proof of the continuity of $Y(t)$, for $t\neq 0$, and hence that of $Y(t,\tau), t\neq\tau$

$$\dot{Y}(t) = A_0 e^{A_0 t} + A_1 e^{A_0(t-h)} + \int_h^t A_0 e^{A_0(t-s)} A_1 e^{A_0(s-h)} ds$$

$$+\sum_{j=2}^{k} \prod_{i\in\{j,j-1,\cdots,2\}} \int_{(i-1)h}^{s_i-h} A_1 e^{A_0(s_i-h-s_{i-1})} A_1 e^{A_0(s_1-h)} \prod_{\lambda=1}^{j} ds_\lambda$$

$$+\left[\sum_{j=2}^{k}\int_{jh}^{t} A_0 e^{A_0(t-s_j)} \prod_{i\in\{j,j-1,\cdots,2\}} \int_{(i-1)h}^{s_i-h} A_1 e^{A_0(s_i-h-s_{i-1})}\right] A_1 e^{A_0(s_1-h)} \prod_{\lambda=1}^{j} ds_\lambda, t\in J_k, k\geq 2$$

$$\Rightarrow \lim_{t\to(kh)^+} \dot{Y}(t) = A_0 e^{A_0 kh} + A_1 e^{A_0[k-1]h} + \int_h^{kh} A_0 e^{A_0(kh-s)} A_1 e^{A_0(s-h)} ds$$

$$+\sum_{j=2}^{k} \prod_{i\in\{j,j-1,\cdots,2\}} \int_{(i-1)h}^{s_i-h} A_1 e^{A_0(s_i-h-s_{i-1})} A_1 e^{A_0(s_1-h)} \prod_{\lambda=1}^{j} ds_\lambda$$

$$+\left[\sum_{j=2}^{k-1}\int_{jh}^{kh} A_0 e^{A_0(t-s_j)} \prod_{i\in\{j,j-1,\cdots,2\}} \int_{(i-1)h}^{s_i-h} A_1 e^{A_0(s_i-h-s_{i-1})}\right] A_1 e^{A_0(s_1-h)} \prod_{\lambda=1}^{j} ds_\lambda, k\geq 3$$

since the integral contribution at $t = kh$ is zero.

$$t\in J_{k-1} \Rightarrow \lim_{t\to(kh)^-} \dot{Y}(t) = A_0 e^{A_0 kh} + A_1 e^{A_0[k-1]h} + \int_h^{kh} A_0 e^{A_0(kh-s)} A_1 e^{A_0(s-h)} ds$$

$$+\sum_{j=2}^{k-1} \prod_{i\in\{j,j-1,\cdots,2\}} \int_{(i-1)h}^{s_i-h} A_1 e^{A_0(s_i-h-s_{i-1})} A_1 e^{A_0(s_1-h)} \prod_{\lambda=1}^{j} ds_\lambda$$

$$+\left[\sum_{j=2}^{k-1}\int_{jh}^{t} A_0 e^{A_0(kh-s_j)} \prod_{i\in\{j,j-1,\cdots,2\}} \int_{(i-1)h}^{s_i-h} A_1 e^{A_0(s_i-h-s_{i-1})}\right] A_1 e^{A_0(s_1-h)} \prod_{\lambda=1}^{j} ds_\lambda, k-1\geq 2$$

We proceed to take the second derivative of $Y(t)$.

$$\ddot{Y}(t) = A_0^2 e^{A_0 t} + A_1 A_0 e^{A_0(t-h)} + A_0 A_1 e^{A_0(t-h)} + \int_h^t A_0^2 e^{A_0(t-s)} A_1 e^{A_0(s-h)} ds$$

$$+\sum_{j=2}^{k} A_0 \prod_{i\in\{j,j-1,\cdots,2\}} \int_{(i-1)h}^{s_i-h} A_1 e^{A_0(s_i-h-s_{i-1})} A_1 e^{A_0(s_1-h)} \prod_{\lambda=1}^{j} ds_\lambda$$

$$+\left[\sum_{j=2}^{k}\int_{jh}^{t} A_0^2 e^{A_0(t-s_j)} \prod_{i\in\{j,j-1,\cdots,2\}} \int_{(i-1)h}^{s_i-h} A_1 e^{A_0(s_i-h-s_{i-1})}\right] A_1 e^{A_0(s_1-h)} \prod_{\lambda=1}^{j} ds_\lambda, t\in J_k, k\geq 2$$

$$\Rightarrow \lim_{t\to(kh)^+} \ddot{Y}(t) = A_0^2 e^{A_0 kh} + A_1 A_0 e^{A_0[k-1]h} + A_0 A_1 e^{A_0[k-1]h} + \int_h^{kh} A_0^2 e^{A_0(kh-s)} A_1 e^{A_0(s-h)} ds$$

$$+\sum_{j=2}^{k} A_0 \prod_{i\in\{j,j-1,\cdots,2\}} \int_{(i-1)h}^{s_i-h} A_1 e^{A_0(s_i-h-s_{i-1})} A_1 e^{A_0(s_1-h)} \prod_{\lambda=1}^{j} ds_\lambda$$

$$+\left[\sum_{j=2}^{k}\int_{jh}^{kh} A_0^2 e^{A_0(kh-s_j)} \prod_{i\in\{j,j-1,\cdots,2\}} \int_{(i-1)h}^{s_i-h} A_1 e^{A_0(s_i-h-s_{i-1})}\right] A_1 e^{A_0(s_1-h)} \prod_{\lambda=1}^{j} ds_\lambda, k\geq 2$$

$$\lim_{t\to(kh)^-}\ddot{Y}(t) = A_0^2 e^{A_0 kh} + A_1 A_0 e^{A_0[k-1]h} + A_0 A_1 e^{A_0[k-1]h} + \int_h^{kh} A_0^2 e^{A_0(kh-s)} A_1 e^{A_0(s-h)} ds$$

$$+\sum_{j=2}^{k-1} A_0 \prod_{i\in\{j,j-1,\cdots,2\}} \int_{(i-1)h}^{s_i-h} A_1 e^{A_0(s_i-h-s_{i-1})} A_1 e^{A_0(s_1-h)} \prod_{\lambda=1}^{j} ds_\lambda$$

$$+\left[\sum_{j=2}^{k-1}\int_{jh}^{kh} A_0^2 e^{A_0(kh-s_j)} \prod_{i\in\{j,j-1,\cdots,2\}} \int_{(i-1)h}^{s_i-h} A_1 e^{A_0(s_i-h-s_{i-1})}\right] A_1 e^{A_0(s_1-h)} \prod_{\lambda=1}^{j} ds_\lambda, k\geq 2$$

Clearly, $\lim_{t\to(kh)^+}\ddot{Y}(t) = \lim_{t\to(kh)^-}\ddot{Y}(t) + A_0 \prod_{i\in\{k,k-1,\cdots,2\}} \int_{(i-1)h}^{s_i-h} A_1 e^{A_0(s_i-h-s_{i-1})} A_1 e^{A_0(s_1-h)} \prod_{\lambda=1}^{k} ds_\lambda$

Clearly, $\lim_{t\to h^-}\ddot{Y}(t) = A_0^2 e^{A_0 h}$,

$$\ddot{Y}(t) = A_0^2 e^{A_0 t} + A_1 A_0 e^{A_0(t-h)} + A_0 A_1 e^{A_0(t-h)} + \int_h^t A_0^2 e^{A_0(t-s)} A_1 e^{A_0(s-h)} ds$$

$$\Rightarrow \lim_{t\to h^+}\ddot{Y}(t) = A_0^2 e^{A_0 h} + A_1 A_0 + A_0 A_1 \Rightarrow \lim_{t\to h^+}\ddot{Y}(t) = \lim_{t\to h^-}\ddot{Y}(t) + A_1 A_0 + A_0 A_1$$

Therefore, $Y(t)$ is differentiable except at 0, but not twice differentiable at kh, for any nonnegative integer k. This completes the proof that $Y(t)$ is non-analytic at kh, for k nonnegative.

Additional information revealed is the differentiability of $Y(t)$, for $t \neq 0$.

We conclude the following:

(i) $Y(t,\tau)$ is analytic except at $t=\tau+kh, k\in\{0,1,\cdots\} \Rightarrow Y(t,\tau)$ is analytic for $t\in\mathbf{R}\setminus\{\tau+kh : k=0,1,\cdots\}$

(ii) $Y(t,\tau)$ is differentiable at $t=\tau+kh, k\neq 0.$

$X(\tau,t_1)=e^{A_0(t_1-\tau)}$, for $\tau\in K_0 \Rightarrow \lim_{\tau\to t_1^-} X(\tau,t_1)=I_n$; $\lim_{\tau\to t_1^+} X(\tau,t_1)=0 \Rightarrow X(\tau,t_1)$ is not continuous at $\tau=t_1$ and hence not analytic there. $\lim_{\tau\to(t_1-h)^+} X(\tau,t_1)=e^{A_0 h}$,

$$\tau\in K_1 \Rightarrow X(\tau,t_1)= e^{A_0(t_1-\tau)} - \int_{t_1-h}^{\tau} e^{A_0(t_1-h-s_1)} A_1 e^{A_0(s_1-\tau)} ds_1 \Rightarrow \lim_{\tau\to(t_1-h)^-} X(\tau,t_1)=e^{A_0 h}.$$

Therefore $X(\tau,t_1)$ is continuous at $\tau=t_1-h.$ By Leibniz's rule of integral differentiation,

$$\tau\in K_1 \Rightarrow \dot{X}(\tau,t_1)= A_0 e^{A_0(t_1-\tau)} - e^{A_0(t_1-h-\tau)} A_1 + 0 - \int_{t_1-h}^{\tau} e^{A_0(t_1-h-s_1)} A_1 A_0 e^{A_0(s_1-\tau)} ds_1$$

$$\Rightarrow \lim_{\tau\to(t_1-h)^+} X(\tau,t_1)=A_0 e^{A_0 h} - A_1;\ \lim_{\tau\to(t_1-h)^+}\dot{X}(\tau,t_1)=-A_0 e^{A_0 h}.$$

Therefore $X(\tau,t_1)$ is not differentiable at $\tau=t_1-h,$ and hence not analytic there.

$$\frac{\partial}{\partial\tau}X(\tau,t_1)=-A_0e^{A_0(t_1-\tau)}-e^{A_0(t_1-h-s_1)}A_1+\int_{t_1-h}^{\tau}e^{A_0(t_1-h-s_1)}A_1A_0e^{A_0(s_1-\tau)}ds_1$$

$$+\left[\sum_{k=2}^{j}(-1)^j e^{A_0(t_1-h-s_1)}\prod_{i\in\{k,k-1,\cdots,2\}}\int_{(i-1)h}^{s_i-h}\left[\prod_{q=1}^{k-1}A_1e^{A_0(s_q-h-s_{q+1})}\right]A_1\prod_{\lambda=1}^{k}ds_\lambda\right]$$

$$+\left[\sum_{k=2}^{j}(-1)^{j+1}\int_{t_1-kh}^{\tau}e^{A_0(t_1-h-s_1)}\prod_{i\in\{k,k-1,\cdots,2\}}\int_{(i-1)h}^{s_i-h}\left[\prod_{q=1}^{k-1}A_1e^{A_0(s_q-h-s_{q+1})}\right]A_1A_0e^{A_0(s_k-\tau)}\right]\prod_{\lambda=1}^{k}ds_\lambda,\tau\in K_j,\ j\geq 2$$

$$\lim_{\tau\to(t_1-jh)^+}\left[\frac{\partial}{\partial\tau}X(\tau,t_1)\right]=-A_0e^{A_0jh}-e^{A_0[j-1]h}A_1+\int_{t_1-h}^{t_1-jh}e^{A_0(t_1-h-s_1)}A_1A_0e^{A_0(s_1+jh-t_1)}ds_1$$

$$+\left[\sum_{k=2}^{j}(-1)^j e^{A_0(t_1-h-s_1)}\prod_{i\in\{k,k-1,\cdots,2\}}\int_{(i-1)h}^{s_i-h}\left[\prod_{q=1}^{k-1}A_1e^{A_0(s_q-h-s_{q+1})}\right]A_1\prod_{\lambda=1}^{k}ds_\lambda\right]$$

$$+\left[\sum_{k=2}^{j-1}(-1)^{j+1}\int_{t_1-kh}^{t_1-jh}e^{A_0(t_1-h-s_1)}\prod_{i\in\{k,k-1,\cdots,2\}}\int_{(i-1)h}^{s_i-h}\left[\prod_{q=1}^{k-1}A_1e^{A_0(s_q-h-s_{q+1})}\right]A_1A_0e^{A_0(s_k+jh-t_1)}\right]\prod_{\lambda=1}^{k}ds_\lambda,\ j-1\geq 2$$

since the integral contribution from $k=j$ is zero.

$$\tau\in K_{j-1}\Rightarrow\lim_{\tau\to(t_1-jh)^-}\left[\frac{\partial}{\partial\tau}X(\tau,t_1)\right]=-A_0e^{A_0jh}-e^{A_0[j-1]h}A_1+\int_{t_1-h}^{t_1-jh}e^{A_0(t_1-h-s_1)}A_1A_0e^{A_0(s_1+jh-t_1)}ds_1$$

$$+\left[\sum_{k=2}^{j}(-1)^j e^{A_0(t_1-h-s_1)}\prod_{i\in\{k,k-1,\cdots,2\}}\int_{(i-1)h}^{s_i-h}\left[\prod_{q=1}^{k-1}A_1e^{A_0(s_q-h-s_{q+1})}\right]A_1\prod_{\lambda=1}^{k}ds_\lambda\right]$$

$$+\left[\sum_{k=2}^{j-1}(-1)^{j+1}\int_{t_1-kh}^{t_1-jh}e^{A_0(t_1-h-s_1)}\prod_{i\in\{k,k-1,\cdots,2\}}\int_{(i-1)h}^{s_i-h}\left[\prod_{q=1}^{k-1}A_1e^{A_0(s_q-h-s_{q+1})}\right]A_1A_0e^{A_0(s_k+jh-t_1)}\right]\prod_{\lambda=1}^{k}ds_\lambda,\ j-1\geq 2$$

Therefore $X(\tau,t_1)$ is differentiable except at $\tau=t_1$. We proceed to take the second derivative:

$$\frac{\partial^2}{\partial\tau^2}X(\tau,t_1)=A_0^2e^{A_0(t_1-\tau)}+e^{A_0(t_1-h-\tau)}A_1A_0-\int_{t_1-h}^{\tau}e^{A_0(t_1-h-s_1)}A_1A_0^2e^{A_0(s_1-\tau)}ds_1$$

$$+\left[\sum_{k=2}^{j}(-1)^{j+1}e^{A_0(t_1-h-s_1)}\prod_{i\in\{k,k-1,\cdots,2\}}\int_{(i-1)h}^{s_i-h}\left[\prod_{q=1}^{k-1}A_1e^{A_0(s_q-h-s_{q+1})}\right]A_1A_0\prod_{\lambda=1}^{k}ds_\lambda\right]\prod_{\lambda=1}^{k}ds_\lambda$$

$$+\left[\sum_{k=2}^{j}(-1)^{j+2}\int_{t_1-kh}^{\tau}e^{A_0(t_1-h-s_1)}\prod_{i\in\{k,k-1,\cdots,2\}}\int_{(i-1)h}^{s_i-h}\left[\prod_{q=1}^{k-1}A_1e^{A_0(s_q-h-s_{q+1})}\right]A_1A_0^2e^{A_0(s_k-\tau)}\right]\prod_{\lambda=1}^{k}ds_\lambda,\tau\in K_j,\ j\geq 2$$

$$\Rightarrow\lim_{\tau\to(t_1-jh)^+}\left[\frac{\partial^2}{\partial\tau^2}X(\tau,t_1)\right]=A_0^2e^{A_0jh}+e^{A_0[j-1]h}A_1A_0-\int_{t_1-h}^{t_1-jh}e^{A_0(t_1-h-s_1)}A_1A_0^2e^{A_0(s_1+jh-t_1)}ds_1$$

$$+\left[\sum_{k=2}^{j}(-1)^{j+1}e^{A_0(t_1-h-s_1)}\prod_{i\in\{k,k-1,\cdots,2\}}\int_{(i-1)h}^{s_i-h}\left[\prod_{q=1}^{k-1}A_1e^{A_0(s_q-h-s_{q+1})}\right]A_1A_0\prod_{\lambda=1}^{k}ds_\lambda\right]\prod_{\lambda=1}^{k}ds_\lambda$$

$$+\left[\sum_{k=2}^{j-1}(-1)^{j+2}\int_{t_1-kh}^{t_1-jh}e^{A_0(t_1-h-s_1)}\prod_{i\in\{k,k-1,\cdots,2\}}\int_{(i-1)h}^{s_i-h}\left[\prod_{q=1}^{k-1}A_1e^{A_0(s_q-h-s_{q+1})}\right]A_1A_0^2e^{A_0(s_k+jh-t_1)}\right]\prod_{\lambda=1}^{k}ds_\lambda,\ j-1\geq 2$$

because the integral contribution from $k = j$ is zero.

$$\tau \in K_{j-1} \Rightarrow \lim_{\tau \to (t_1 - jh)^-} \left[\frac{\partial^2}{\partial \tau^2} X(\tau, t_1) \right] = A_0^2 e^{A_0 jh} + e^{A_0 [j-1]h} A_1 A_0 - \int_{t_1-h}^{t_1-jh} e^{A_0(t_1-h-s_1)} A_1 A_0^2 e^{A_0(s_1+jh-t_1)} ds_1$$

$$+ \left[\sum_{k=2}^{j-1} (-1)^{j+1} e^{A_0(t_1-h-s_1)} \prod_{i \in \{k,k-1,\cdots,2\}} \int_{(i-1)h}^{s_i-h} \left[\prod_{q=1}^{k-1} A_1 e^{A_0(s_q-h-s_{q+1})} \right] A_1 A_0 \prod_{\lambda=1}^{k} ds_\lambda \right] \prod_{\lambda=1}^{k} ds_\lambda$$

$$+ \left[\sum_{k=2}^{j-1} (-1)^{j+2} \int_{t_1-kh}^{t_1-jh} e^{A_0(t_1-h-s_1)} \prod_{i \in \{k,k-1,\cdots,2\}} \int_{(i-1)h}^{s_i-h} \left[\prod_{q=1}^{k-1} A_1 e^{A_0(s_q-h-s_{q+1})} \right] A_1 A_0^2 e^{A_0(s_k+jh-t_1)} \right] \prod_{\lambda=1}^{k} ds_\lambda, \; j-1 \geq 2$$

Hence $\lim_{\tau \to (t_1 - jh)^+} X(\tau, t_1) \neq \lim_{\tau \to (t_1 - jh)^+} X(\tau, t_1), j \geq 2 : t_1 - [j+1]h \geq 0.$

Therefore, $X(\tau, t_1)$ is differentiable except at $\tau = t_1$, but not twice differentiable at $\tau = t_1 - jh$, for any nonnegative integer j. This completes the proof that $X(\tau, t_1)$ is not analytic at $\tau = t_1 - jh$, for any nonnegative integer j.

Additional information revealed is the differentiability of $X(\tau, t_1)$, for $\tau \neq t_1$.

We conclude the following:

(i) $X(\tau, t_1)$ is analytic except at $\tau = t_1 - jh, j \in \{0,1,\cdots\} : t_1 - [j+1]h \geq 0$

$\Rightarrow X(\tau, t_1)$ is analytic for $\tau \in \mathbf{R} \setminus \{t_1 - jh : j = 0,1,\cdots\}$

(ii) $X(\tau, t_1)$ is differentiable at $\tau = t_1 - jh, j \neq 0.$

Proof of (viii)
Given (14), (15) and (16), observe that

$\psi(t_1) = \phi^{\mathrm{T}}(0)$ and $\psi(t_1 - h) = \phi^{\mathrm{T}}(-h)$; t_1 is the terminal point for ψ, while 0 is the initial point for ϕ; $t_1 - h$ is the initial point for ψ, while $-h$ is the initial point for ϕ.

The variation of constants formula for the initial function problem (1) and (2) is:

$$x(t) = Y(t)\phi(0) + \int_{-h}^{0} Y(t-h-s) A_1 \phi(s) ds, \; t \geq 0 \tag{17}$$

Cf. Chukwu (1992, pp. 125-126, 345) and Manitius (1978, pp. 6, 12) , while the variation of constants formula for the terminal function problem (14) and (15) is:

$$y(\tau) = \phi(0) X(t_1 - \tau) - \int_{t_1}^{\tau} \phi(t_1 - h - s) A_1 X(s - h - \tau) d\tau, \; \tau \geq 0 \tag{18}$$

This completes the proof that the stated initial and terminal function problems are duals of each other.

VI. CONCLUSION

This article has established the duality between the solution matrices in Ukwu and Garba [2014a] and the corresponding indices of control systems in Ukwu and Garba [2014c], as well as the duality between their variation of constants formulas, culminating in the conclusion that the associated initial and terminal function problems are duals of each other. The proofs were achieved using an algorithm involving deft application of rules of integral differentiation and key transformations from one problem to the other, leading to the assertion that the results from one problem can be realized from the other. The ideas exposed in this paper can be exploited to extend the results to double-delay and neutral systems if the structures of the associated solution and control index matrices can be determined.

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