International Journal of Mathematics and Statics Studies

Vol.2, No.2, pp.28-42, June 2014

Published by European Centre for Research Training and Development UK (www.ea-journals.org)

THE STRUCTURE OF INDICES OF CONTROL SYSTEMS FOR CERTAIN SINGLE – DELAY AUTONOMOUS LINEAR SYSTEMS WITH PROBLEM INSTANCES

Ukwu Chukwunenye

Department of Mathematics, University of Jos, P.M.B 2084, Jos, Postal Code: 930001, Plateau State, Nigeria.

ABSTRACT: *This paper investigated the structure of indices of control systems for single – delay autonomous linear systems on the interval [0, 4h] and on [0, ∞) for special coefficient matrix cases. It also provided a note on Euclidean controllability and application instances of the determination of their controllability dispositions. The development of the associated control index matrices exploited the continuity of these matrices for positive time periods, change of variables technique, the method of steps and backward continuation recursions to obtain these matrices on successive intervals of length equal to the delay h.The indices were derived using the stage – wise algorithmic format, starting from the right - most interval of length h. The structure could be gleaned and deciphered from the emerged sign convention and recognizable exponential integral ordering.*

KEYWORDS: Controllability, Index, Matrices, Method, Recursive, Steps.

INTRODUCTION

Controllability results for multifarious and specific types of hereditary systems with diversity in treatment approaches are quite prevalent in control literature. Bank and Kent (1972) discussed Controllability of functional differential equations of retarded and neutral types to targets in function space; Jacobs and Langenhop (1976) obtained some criteria for function space controllability of linear neutral systems ; Dauer and Gahl (1977) looked at controllability of nonlinear delay systems; Angell (1980) discussed controllability of nonlinear hereditary systems, using a fixed-point approach; Onwuatu (1984) studied null controllability in function space of nonlinear neutral differential systems with limited controls; Balachandran (1986) discussed controllability of nonlinear systems with delays in both state and control using a constructive control approach and an appeal to Arzela-Ascoli, and Shauder fixed point theorems to guarantee the existence and admissibility of such controls; Underwood and Chukwu (1988) investigated null controllability of nonlinear neutral differential equations, Balachandran(1992) , Balachandran and Balasubramaniam (1993) studied controllability of Volterra Integro-differential systems;

In recent years, Chukwu (2001) formulated differential models and neutral systems for controlling the Wealth of Nations. His monograph derives from economic principles of the dynamics of national income, interest rate, employment, value of capital stock, prices and cumulative balances of payments. Chukwu used a Volterra neutral integro-differential game of pursuit where the quarry control is government intervention in the form of taxation, control of money and supply tariffs. Other relevant works by Chukwu in this area include Chukwu (2002) on Stability and time-optimal control of hereditary systems with application to the economic dynamics of the US, Chukwu (2003).

More research efforts on controllability include Iheagwam (2003), where the author investigated the properties of cores for which the system with distributed delays in control is relatively controllable; Balachandran and Anandhi (2004); Davies and Jackreece (2005); where the authors established sufficient conditions for the controllability and null controllability of linear systems; Other notable results with focus on integro-differential equations and impulsive differential equations with finite and infinite delays include Chang and Chalishajar (2008), Chang et al (2009), Balachandran and Annapoorani (2009), Ye (2010), Ji et al (2011), Vijayakumar et al (2011, 2012), Selvi and Mallika (2012). Some author's established sufficient conditions for the controllability and null controllability of linear systems using the variation of constant formula to deduce their controllability Grammian and exploiting the properties of the Grammian and the asymptotic stability of the free system, Machado et al (2013). These works and others appropriate relevant Existence and Uniqueness of solutions theorems; the linear systems among the cited works use the qualitative properties of the indices of control systems or rank conditions to characterize controllability for the most part. The expressions for such indices were not determined.

The importance of indices of control systems matrices derives from the fact that they not only pave the way for the derivation of determining matrices for the determination of Euclidean controllability and compactness of cores of Euclidean targets but can be used independently for such determination. In sharp contrast to determining matrices the use of indices of control systems for the investigation of the Euclidean controllability of systems can be quite computationally challenging; however this difficulty can be mitigated if the coefficient matrix associated with the state variable at time t is diagonal. This paper pioneers the development of the structure of these indices, with illustrative examples as they relate to Euclidean controllability.

THEORETICAL UNDERPINNING

Literature on state space approach to control studies is replete with indices of control systems as key components for the investigation of controllability. See Gabsov and Kirillova (1976), Manitius (1978), Tadmore (1984), Ukwu (1987), Chukwu (1992), and Ukwu (1992, 1996). Regrettably no other author has made any attempt to obtain general expressions for the associated matrices or special cases of such matrices involving the double - delay h and $2h$. Effort is usually focused on the single – delay mode with the usual approach being to start from the interval $[t_1 - h, t_1]$ and compute the index matrices for given problem instances; then the method of steps and backward continuation recursive procedure are deployed to extend these to the intervals $[t_1 - (k+1)h,\ t_1 - kh]$, for positive integral k, not exceeding 2, for the most part. Such approach is rather restrictive and doomed to failure in terms of structure for arbitrary k. In other words such approach fails to address the issue of the structure of control index matrices. The need to address such short-comings has become imperative; this is the major contribution of this paper, with its wide-ranging implications for extensions to more general systems and holistic approach to controllability studies.

METHODOLOGY

Consider the system:

$$\frac{\partial}{\partial \tau} X(\tau,t) = -X(\tau,t)A_0 - X(\tau+h,t)A_1 \quad (1)$$

for $0<\tau<t, \tau \neq t-kh, k=0,1,...$ where

$$X(\tau, t) = \begin{cases} I_n; \tau = t \\ 0; \ \tau > t \end{cases} \quad (2)$$

A_0, A_1 are $n\times n$ constant matrices and $\tau \to X(\tau,t), \tau \to X(\tau, t+h)$ are $n\times n$ matrix functions. See Chukwu (1992), Hale (1977) and Tadmore (1984) for properties of $X(t,\tau)$. Of particular importance is the fact that $\tau \to X(\tau,t)$ is analytic on the intervals $(t_1-(j+1)h, t_1-jh)$, $j=0,1,...;$ $t_1-(j+1)h>0$. Any such $\tau \in (t_1-(j+1)h, t_1-jh)$ is called a regular point of $\tau \to X(t,\tau)$.

Definition 1: Index of control systems

The expression $c^* X(\tau,t_1)B$ is called the index of a given control system, where c is an n-dimensional constant column vector, $X(\tau,t_1)$ is defined in (1), B is an $n\times m$ constant matrix associated with the control system:

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + Bu(t)$$

and $u(.)$ is an m-vector admissible control function. Thus the control index matrix, $X(\tau,t_1)$ determines the structure of the index of a given control system.

Definition 2: Euclidean Controllability

System (1) is said to be Euclidean controllable on the interval $[0, t_1]$, if for each ϕ in $C([-2h, 0], E^n)$ and $x_1 \in E^n$, there is an admissible control $u \in L_\infty([0,t_1], E^n)$ such that $x_0(\phi,u) = \phi$ and $x(t_1,\phi,u) = x_1$. System (1.1) is Euclidean controllable if it is Euclidean controllable on every interval $[0, t_1]$, $t_1>0$.

A note on Euclidean controllability

$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + Bu(t)$ is Euclidean controllable on $[0, t_1]$ if and only if $c^* X(\tau,t_1)B = 0 \Rightarrow c=0$, for any $c \in \mathbf{R}^n$. See Ukwu (1987, 1992, 1996).

It is important to note that that the Euclidean controllability of a system on an interval $[0,t_1], t_1 \in [qh,(q+1)h)$ for some positive integer q is equivalent to its

Euclidean on controllability on some sub-interval of $[0,t_1]$, of length h. Thus if a system fails to be Euclidean controllable on $[0,t_1]$, it cannot be controllable on any sub - interval of $[0,t_1]$. On the other hand a system may fail to be Euclidean controllable on an interval $[0,t_1]$, but turns out to be Euclidean controllable on an interval $[t_1, t_1 + jh]$, for some positive integer j. A system is Euclidean controllable on an interval if it is Euclidean controllable on any of its sub-intervals.

We proceed to determine the structure of the above control index matrices. This will be achieved using the method of steps and a Backward Continuation Recursive procedure.

RESULTS

Theorem 1

Let $K_j = \left[t_1 - (j+1)h, t_1 - jh\right], \forall\, j : t_1 - (j+1)h > 0$, and fixed $t_1 > 0$.

The control index matrices are defined on successive sub-intervals of $[t_1 - 4h, t_1]$ of length h by

$$\Rightarrow X(\tau, t_1) = \begin{cases} e^{A_0(t_1-\tau)}, \tau \in K_0; \\ e^{A_0(t_1-\tau)} - \int\limits_{t_1-h}^{\tau} e^{A_0(t_1-h-s_1)} A_1 e^{A_0(s_1-\tau)} ds_1,\ \tau \in K_1; \\ e^{A_0(t_1-\tau)} - \int\limits_{t_1-h}^{\tau} e^{A_0(t_1-h-s_1)} A_1 e^{A_0(s_1-\tau)} ds_1 + \int\limits_{t_1-2h}^{\tau} \int\limits_{t_1-h}^{s_2+h} e^{A_0(t_1-h-s_1)} A_1 e^{A_0(s_1-h-s_2)} A_1 e^{A_0(s_2-\tau)} ds_1 ds_2 \\ \text{for } \tau \in K_2; \\ e^{A_0(t_1-\tau)} - \int\limits_{t_1-h}^{\tau} e^{A_0(t_1-h-s_1)} A_1 e^{A_0(s_1-\tau)} ds_1 + \int\limits_{t_1-2h}^{\tau} \int\limits_{t_1-h}^{s_2+h} e^{A_0(t_1-h-s_1)} A_1 e^{A_0(s_1-h-s_2)} A_1 e^{A_0(s_2-\tau)} ds_1 ds_{,2} \\ - \int\limits_{t_1-3h}^{\tau} \int\limits_{t_1-2h}^{s_3+h} \int\limits_{t_1-h}^{s_2+h} e^{A_0(t_1-h-s_1)} A_1 e^{A_0(s_1-h-s_2)} A_1 e^{A_0(s_2-h-s_3)} A_1 e^{A_0(s_3-\tau)} ds_1 ds_2 ds_3, \tau \in K_3. \end{cases}$$

Stage 1

Consider the τ-interval $[t_1 - h,\ t_1]$. Then $\tau + h \in [t_1,\ t_1 + h] \Rightarrow X(\tau + h, t_1) = 0$ on $(t_1,\ t_1 + h]$.

$$\frac{\partial}{\partial\tau}\left[X(\tau,t)e^{A_0(\tau-t)}\right] = \left[\frac{\partial}{\partial\tau}X(\tau,t) + X(\tau,t)A_0\right]e^{A_0(\tau-t)} = -X(\tau+h,t)A_1e^{A_0(\tau-t)}, X(\tau+h,t) = 0\ a.e$$

$\Rightarrow X(\tau,t_1)e^{A_0(\tau-t_1)} = C$, where C is some constant matrix. $\tau = t_1 \Rightarrow C = I_n$, using (2).

Therefore

$$X(\tau,t_1) = e^{A_0(t_1-\tau)}, \text{ for } \tau \in (t_1 - h,\ t_1). \tag{3}$$

Observe that $X(t_1^-, t_1) = I_n$. We will continue to rely on the relation

$$\frac{\partial}{\partial \tau}\left[X(\tau,t)e^{A_0(\tau-t)}\right] = -X(\tau+h,t)A_1e^{A_0(\tau-t)} \tag{4}$$

for the rest of this investigation. The process is terminated as soon as $t_1 - (j+1) < 0$, for some non-negative integer j. If $t_1 - 2h < 0$, STOP. Else proceed to the next stage.

Stage 2 Consider the τ-interval $[t_1 - 2h,\ t_1 - h]$. Then $\tau + h \in [t_1 - h,\ t_1] \Rightarrow X(\tau+h,t_1) = e^{A_0(t_1-h-\tau)}$ (5)

$$\text{(4) and (5)} \Rightarrow \frac{\partial}{\partial \tau}\left[X(\tau,t_1)e^{A_0(\tau-t_1)}\right] = -e^{A_0(t_1-h-\tau)}A_1e^{A_0(\tau-t_1)} \tag{6}$$

$$\Rightarrow X(\tau,t_1)e^{A_0(\tau-t_1)} - X(t_1-h,t_1)e^{A_0(t_1-h-t_1)} = -\int_{t_1-h}^{\tau} e^{A_0(t_1-h-s_1)}A_1e^{A_0(s-t_1)}ds_1 \tag{7}$$

$$= \int_{\tau}^{t_1-h} e^{A_0(s_1-\tau)}A_1e^{A_0(s_1+h-\tau)}ds_1, \ \left(\text{using } \sigma = \tau - s_1 + t_1 - h;\ \text{then } \sigma \to s_1\right) \tag{8}$$

$$\text{Clearly, } X(t_1-h,t_1) = e^{A_0h} \Rightarrow X(t_1-h,t_1)e^{A_0(t_1-h-t_1)} = I_n \tag{9}$$

$$\Rightarrow X(\tau,t_1) = e^{A_0(t_1-\tau)} + \int_{\tau}^{t_1-h} e^{A_0(s_1-\tau)}A_1e^{A_0(\tau-h-s_1)}e^{A_0(t_1-\tau)}ds_1 \tag{10}$$

$$\Rightarrow X(\tau,t_1) = e^{A_0(t_1-\tau)} + \int_{\tau}^{t_1-h} e^{A_0(s_1-\tau)}A_1e^{A_0(t_1-h-s_1)}ds_1 \tag{11}$$

$$\text{or } \ X(\tau,t_1) = e^{A_0(t_1-\tau)} - \int_{t_1-h}^{\tau} e^{A_0(t_1-h-s_1)}A_1e^{A_0(s_1-\tau)}ds_1 \tag{12}$$

Observe that $X((t_1-h)^-,t_1) = X((t_1-h)^+,t_1) = e^{A_0h}$; Therefore $\Delta X(t_1-h,t_1) = 0$.

$\tau \in [t_1-(j+1)h,\ t_1 - jh] \Rightarrow \ X(\tau,t_1)e^{A_0(\tau-t_1)} - X(t_1 - jh,\ t_1)e^{-jA_0h}$

$$= -\int_{t_1-jh}^{\tau} X(s_j+h, A_1e^{A_0(s_j-t_1)}ds_j \tag{13}$$

$$\Rightarrow \ X(\tau,t_1) = X(t_1-jh,\ t_1)e^{A_0(t_1-jh-\tau)} - \int_{t_1-jh}^{\tau} X(s_j+h,\ t_1)A_1e^{A_0(s_j-\tau)}ds_j \tag{14}$$

If $t_1 - 3h < 0$, STOP. Else proceed to the next stage.

Stage 3

Consider the τ-interval K_2. Then $\tau + h \in K_1 \Rightarrow X(\tau+h,t_1)$ and $X(t_1-2h,t_1)$ are applicable on K_1

$$\Rightarrow X(\tau+h,t_1) = e^{A_0(t_1-\tau-h)} + \int_{\tau+h}^{t_1-h} e^{A_0(s_1-\tau-h)}A_1e^{A_0(t_1-h-s_1)}ds_1, \tag{15}$$

$$X(t_1-2h,t_1)=e^{2A_0h}-\int_{t_1-h}^{t_1-2h} e^{A_0(t_1-h-s_1)}A_1e^{A_0(s_1+2h-t_1)}ds_1\,, \tag{16}$$

$$\Rightarrow X(\tau,t_1)=X(t_1-2h,t_1)e^{A_0(t_1-2h-\tau)}-\int_{t_1-2h}^{\tau} X(s_2+h,t_1)A_1e^{A_0(s_2-\tau)}ds_1 \tag{17}$$

$$\Rightarrow X(\tau,t_1)=e^{A_0(t_1-\tau)}-\int_{t_1-h}^{t_1-2h} e^{A_0(t_1-h-s_1)}A_1e^{A_0(s_1-\tau)}ds_1-\int_{t_1-2h}^{\tau} e^{A_0(t_1-h-s_1)}A_1e^{A_0(s_1-\tau)}ds_1 \tag{18}$$

$$-\int_{t_1-2h}^{\tau}\int_{s_2+h}^{t_1-h} e^{A_0(s_1-h-s_2)}A_1e^{A_0(t_1-h-s_1)}A_1e^{A_0(s_2-\tau)}ds_1ds_2 \tag{19}$$

$$\Rightarrow X(\tau,t_1)=e^{A_0(t_1-\tau)}-\int_{t_1-h}^{\tau} e^{A_0(t_1-h-s_1)}A_1e^{A_0(s_1-\tau)}ds_1$$

$$-\int_{t_1-2h}^{\tau}\int_{s_2+h}^{t_1-h} e^{A_0(s_1-h-s_2)}A_1e^{A_0(t_1-h-s_1)}A_1e^{A_0(s_2-\tau)}ds_1ds_2 \tag{20}$$

We preserve the single integral in (20) and transform the double integral to

$$+\int_{t_1-2h}^{\tau}\int_{t_1-h}^{s_2+h} e^{A_0(t_1-h-s_1)}A_1e^{A_0(s_1-h-s_2)}A_1e^{A_0(s_2-\tau)}ds_1ds_2,\ \text{ using } \sigma_1=s_2+h-s_1+t_1-h \tag{21}$$

Hence

$$X(\tau,t_1)=e^{A_0(t_1-\tau)}-\int_{t_1-h}^{\tau} e^{A_0(t_1-h-s_1)}A_1e^{A_0(s_1-\tau)}ds_1$$

$$+\int_{t_1-2h}^{\tau}\int_{t_1-h}^{s_2+h} e^{A_0(t_1-h-s_1)}A_1e^{A_0(s_1-h-s_2)}A_1e^{A_0(s_2-\tau)}ds_1ds_2 \tag{22}$$

By (11) and (21), $X((t_1-2h)^+,\ t_1)=X((t_1-2h)^-,\ t_1)$

$$=e^{2A_0h}+\int_{t_1-2h}^{t_1-h} e^{A_0(s+2h-t_1)}A_1e^{A_0(t_1-h-s_1)}ds_1 \tag{23}$$

Hence $\Delta X(t_1-2h,\ t_1)=0$, where $\Delta X(t_1-2h,\ t_1)=X((t_1-2h)^-,\ t_1)-X((t_1-2h)^+,\ t_1)$

If $t_1-4h\ <\ 0$, STOP. Else proceed to the next stage.

Stage Consider the interval $[t_1-4h,\ t_1-3h]$. Then $\tau+h\in[t_1-3h,\ t_1-2h]$, $X(\tau+h,t_1)$ and $X(t_1-3h,\ t_1)$ are applicable on $[t_1-3h,t_1-2h]$. We invoke (14) with $j=3$ to deduce that

$$X(\tau,t_1)=X(t_1-3h,\ t_1)e^{A_0(t_1-3h-\tau)}-\int_{t_1-3h}^{\tau}X(s_3+h,\ t_1)A_1e^{A_0(s_3-\tau)}ds_3 \qquad (24)$$

$$X(\tau,t_1)=e^{A_0(t_1-\tau)}-\int_{t_1-h}^{\tau}e^{A_0(t_1-h-s_1)}A_1e^{A_0(s_1-\tau)}ds_1$$

$$-\int_{t_1-2h}^{\tau}\int_{s_2+h}^{t_1-h}e^{A_0(s_1-h-s_2)}A_1e^{A_0(t_1-h-s_1)}A_1e^{A_0(s_2-\tau)}ds_1ds_2,\ \text{on } [t_1-3h,t_1-2h] \qquad (25)$$

$$\Rightarrow X(\tau,t_1)\ =\ e^{A_0(t_1-\tau)}-\int_{t_1-h}^{\tau}e^{A_0(t_1-h-s_1)}A_1e^{A_0(s_1-\tau)}ds_1$$

$$+\int_{t_1-2h}^{\tau}\int_{t_1-h}^{s_2+h}e^{A_0(t_1-h-s_1)}A_1e^{A_0(s_1-h-s_2)}A_1e^{A_0(s_2-\tau)}ds_1ds_2,\ \text{on } [t_1-3h,t_1-2h] \qquad (26)$$

Expression (26) has the desired structure, in the sense of alternating signs and recognizable exponential ordering.

$$(20)\Rightarrow X(\tau+h,t_1)=e^{A_0(t_1-h-\tau)}-\int_{t_1-h}^{\tau+h}e^{A_0(t_1-h-s_1)}A_1e^{A_0(s_1-h-\tau)}ds_1$$

$$-\int_{t_1-2h}^{\tau+h}\int_{s_2+h}^{t_1-h}e^{A_0(s_1-h-s_2)}A_1e^{A_0(t_1-h-s_1)}A_1e^{A_0(s_2-h-\tau)}ds_1ds_2 \qquad (27)$$

$$X(t_1-3h,\ t_1)=e^{3A_0h}-\int_{t_1-h}^{t_1-3h}e^{A_0(t_1-h-s_1)}A_1e^{A_0(s_1+3h-t_1)}ds_1$$

$$-\int_{t_1-2h}^{t_1-3h}\int_{s_2+h}^{t_1-h}e^{A_0(s_1-h-s_2)}A_1e^{A_0(t_1-h-s_1)}A_1e^{A_0(s_2+3h-t_1)}ds_1ds_2 \qquad (28)$$

$$\Rightarrow\ X(\tau,t_1)=X(t_1-3h,\ t_1)e^{A_0(t_1-3h-\tau)}-\int_{t_1-3h}^{\tau}X(s_3+h,\ t_1)A_1e^{A_0(s_3-\tau)}ds_3 \qquad (29)$$

$$\Rightarrow X(\tau,\ t_1)=e^{A_0(t_1-\tau)}-\int_{t_1-h}^{t_1-3h}e^{A_0(t_1-h-s_1)}A_1e^{A_0(s_1-\tau)}ds_1 \qquad (30)$$

$$-\int_{t_1-2h}^{t_1-3h}\int_{s_2+h}^{t_1-h} e^{A_0(s_1-h-s_2)}A_1e^{A_0(t_1-h-s_1)}A_1e^{A_0(s_2-\tau)}ds_1ds_2 \tag{31}$$

$$-\int_{t_1-3h}^{\tau} e^{A_0(t_1-h-s_3)}A_1e^{A_0(s_3-\tau)}ds_3+\int_{t_1-3h}^{\tau}\int_{t_1-h}^{s_3+h} e^{A_0(t_1-h-s_1)}A_1e^{A_0(s_1-h-s_3)}A_1e^{A_0(s_3-\tau)}ds_1ds_3 \tag{32}$$

$$+\int_{t_1-3h}^{\tau}\int_{t_1-2h}^{s_3+h}\int_{s_2+h}^{t_1-h} e^{A_0(s_1-h-s_2)}A_1e^{A_0(t_1-h-s_1)}A_1e^{A_0(s_2-h-s_3)}A_1e^{A_0(s_3-\tau)}ds_1ds_2ds_3 \tag{33}$$

Use the change of variables $\sigma_1 = s_2 + h - s_1 + t_1 - h = s_3 - s_1 + t_1$ in (31) to obtain

$$-\int_{t_1-2h}^{t_1-3h}\int_{s_2+h}^{t_1-h} e^{A_0(s_1-h-s_2)}A_1e^{A_0(t_1-h-s_1)}A_1e^{A_0(s_2-\tau)}ds_1ds_2 = \int_{t_1-2h}^{t_1-3h}\int_{t_1-h}^{s_2+h} e^{A_0(t_1-h-s_1)}A_1e^{A_0(s_1-h-s_2)}A_1e^{A_0(s_2-\tau)}ds_1ds_2 \tag{34}$$

Use the change of variables $\sigma_1 = s_2 + h - s_1 + t_1 - h$ to transform the triple integral (33) to get

$$-\int_{t_1-3h}^{\tau}\int_{t_1-2h}^{s_3+h}\int_{t_1-h}^{s_2+h} e^{A_0(t_1-h-s_1)}A_1e^{A_0(s_1-h-s_2)}A_1e^{A_0(s_2-h-s_3)}A_1e^{A_0(s_3-\tau)}ds_1ds_2ds_3 \tag{35}$$

$$\Rightarrow X(\tau, t_1) = e^{A_0(t_1-\tau)} - \int_{t_1-h}^{\tau} e^{A_0(t_1-h-s_1)}A_1e^{A_0(s_1-\tau)}ds_1 + \int_{t_1-2h}^{\tau}\int_{t_1-h}^{s_2+h} e^{A_0(t_1-h-s_1)}A_1e^{A_0(s_1-h-s_2)}A_1e^{A_0(s_2-\tau)}ds_1ds_2$$

$$-\int_{t_1-3h}^{\tau}\int_{t_1-2h}^{s_3+h}\int_{t_1-h}^{s_2+h} e^{A_0(t_1-h-s_1)}A_1e^{A_0(s_1-h-s_2)}A_1e^{A_0(s_2-h-s_3)}A_1e^{A_0(s_3-\tau)}ds_1ds_2ds_3 \tag{36}$$

4.2 Corollary 1

If $A_0 = 0$, then

$$X(\tau,t_1)=\begin{cases} I_n,\ t\in K_0; \\ I_n + A_1(t_1-\tau-h),\ \tau\in K_1; \\ I_n + A_1(t_1-\tau-h)+A_1^2\dfrac{(t_1-\tau-2h)^2}{2!},\tau\in K_2; \\ I_n + A_1(t_1-\tau-h)+A_1^2\dfrac{(t_1-\tau-2h)^2}{2!}+A_1^3\dfrac{(t_1-\tau-3h)^2}{3!},\ \tau\in K_3 \end{cases}$$

4.3 Corollary 2

If $A_1 = \mathrm{diag}(b)$, then

$$X(\tau,t_1)=\begin{cases} e^{A_0(t_1-\tau)},\ \tau\in K_0; \\ e^{A_0(t_1-\tau)}+b(t_1-\tau-h)e^{A_0(t_1-\tau-h)},\ \tau\in K_1; \\ e^{A_0(t_1-\tau)}+b(t_1-\tau-h)e^{A_0(t_1-\tau-h)}+b^2\dfrac{(t_1-\tau-2h)^2}{2!}e^{A_0(t_1-\tau-2h)},\tau\in K_2; \\ e^{A_0(t_1-\tau)}+b(t_1-\tau-h)e^{A_0(t_1-\tau-h)}+b^2\dfrac{(t_1-\tau-2h)^2}{2!}e^{A_0(t_1-\tau-2h)}+b^3\dfrac{(t_1-\tau-3h)^2}{3!}e^{A_0(t_1-\tau-3h)} \\ +b^3\dfrac{(t_1-\tau-3h)^2}{3!}e^{A_0(t_1-\tau-3h)},\ \tau\in K_3 \end{cases}$$

Corollary 1 motivates the following theorem:

Theorem 2

Consider the system (1) with $A_0=\mathrm{diag}(a)$. Then the control index matrices are given by

$$X(\tau,t_1)=\begin{cases} e^{a(t_1-\tau)}I_n,\ \tau\in K_0; \\ e^{a(t_1-\tau)}I_n+\sum_{i=1}^{j}A_1^i\dfrac{(t_1-\tau-ih)^i}{i!}e^{a(t_1-\tau-ih)},\ \tau\in K_j. \end{cases}$$

Proof

The theorem is valid for $j\in\{0,1,2,3\}$, as seen from the last corollary. Assume the validity of the theorem for $\tau\in K_p, 4\le p\le j,$ for some $j\ge 5.$ Then $\tau+h,\ t_1-(j+1)h, s_{j+1}+h\in K_j\Rightarrow$

$$X(t_1-(j+1)h,\ t_1)=e^{a([j+1]h)}I_n+\sum_{i=1}^{j}A_1^i\frac{([j+1-i]h)^i}{i!}e^{a([j+1-i]h)};$$

$$X(\tau+h,t_1)=e^{a(t_1-h-\tau)}I_n+\sum_{i=1}^{j}A_1^i\frac{(t_1-\tau-[i+1]h)^i}{i!}e^{a(t_1-\tau-[i+1]h)}.$$

On K_{j+1},

$$X(\tau,t_1)=X(t_1-(j+1)h,\ t_1)e^{a(t_1-(j+1)h-\tau)}-\int_{t_1-(j+1)h}^{\tau}X(s_{j+1}+h,\ t_1)A_1e^{a(s_{j+1}-\tau)}ds_{j+1}$$

$$=\ e^{a(t_1-\tau)}+\sum_{i=1}^{j}A_1^i\frac{([j+1-i]h)^i}{i!}e^{a(t_1-\tau-ih)}$$

$$-\int_{t_1-(j+1)h}^{\tau}\left[A_1e^{a(t_1-h-\tau)}I_n+\sum_{i=1}^{j}A_1^{i+1}\frac{(t_1-s_{j+1}-[i+1]h)^i}{i!}e^{a(t_1-s_{j+1}-[i+1]h)}\right]ds_{j+1}$$

$$= e^{a(t_1-\tau)}I_n + \sum_{i=1}^{j} A_1^i \frac{([j+1-i]h)^i}{i!} e^{a(t_1-\tau-ih)}$$
$$+ A_1 e^{a(t_1-h-\tau)}\left(t_1-(j+1)h-\tau\right) + \sum_{i=1}^{j} A_1^{i+1} \frac{(t_1-\tau-[i+1]h)^{i+1}}{(i+1)!} e^{a(t_1-\tau-[i+1]h)}$$
$$- \sum_{i=1}^{j} A_1^{i+1} \frac{([j-i]h)^{i+1}}{(i+1)!} e^{a(t_1-\tau-[i+1]h)}$$

$$= e^{a(t_1-\tau)}I_n + \sum_{i=1}^{j} A_1^i \frac{([j+1-i]h)^i}{i!} e^{a(t_1-\tau-ih)}$$
$$+ A_1 e^{a(t_1-h-\tau)}\left(t_1-(j+1)h-\tau\right) + \sum_{i=2}^{j+1} A_1^i \frac{(t_1-\tau-ih)^i}{i!} e^{a(t_1-\tau-ih)}$$
$$- \sum_{i=2}^{j+1} A_1^i \frac{([j+1-i]h)^i}{i!} e^{a(t_1-\tau-ih)}$$
$$= e^{a(t_1-\tau)}I_n + A_1 e^{a(t_1-h-\tau)}\left(t_1-(j+1)h-\tau\right) + A_1(jh)e^{a(t_1-\tau-h)} + \sum_{i=2}^{j+1} A_1^i \frac{(t_1-\tau-ih)^i}{i!} e^{a(t_1-\tau-ih)}$$
$$= e^{a(t_1-\tau)}I_n + A_1 e^{a(t_1-h-\tau)}\left(t_1-h-\tau\right) + \sum_{i=2}^{j+1} A_1^i \frac{(t_1-\tau-ih)^i}{i!} e^{a(t_1-\tau-ih)}$$
$$= e^{a(t_1-\tau)}I_n + \sum_{i=1}^{j+1} A_1^i \frac{(t_1-\tau-ih)^i}{i!} e^{a(t_1-\tau-ih)},$$ completing the proof of the theorem.

Remark 1: The special case $A_0 = 0$.

$$A_0 = 0 \Rightarrow X(\tau, t_1) = \begin{cases} I_n, \tau \in K_0; \\ I_n + \sum_{i=1}^{j} A_1^i \frac{(t_1-\tau-ih)^i}{i!}, \ \tau \in K_j, j \geq 1. \end{cases}$$

Remark 2: For the special case $A_0 = 0$ and A_1 nilpotent of index $p < j$,

$$X(\tau, t_1) = \begin{cases} I_n, \tau \in K_0; \\ I_n + \sum_{i=1}^{p-1} A_1^i \frac{(t_1-\tau-ih)^i}{i!}, \ \tau \in K_j, j \geq 1. \end{cases}$$

Corollary 2 motivates the following theorem:

Theorem 3

If $A_1 = \mathrm{diag}(b)$, then

$$X(\tau,t_1)=\begin{cases} e^{A_0(t_1-\tau)},\ t\in K_0; \\ e^{A_0(t_1-\tau)}+\sum_{i=1}^{j} b^i \dfrac{(t_1-\tau-ih)^i}{i!} e^{A_0(t_1-\tau-ih)},\ \tau\in K_j,\ j\geq 1 \end{cases}$$

Proof

The theorem is valid for $j\in\{0,1,2,3\}$, as seen from corollary 2. Simply replace a by A_0 and A_1 by bI_n in theorem 2 and its proof to complete the proof of theorem 3.

IMPLICATION TO RESEARCH AND PRACTICE

Three major tools are used in the investigation of the controllability of autonomous linear hereditary systems: (i) appropriate rank condition on the determining matrices (ii) the construction of the optimal control for transfer of points (iii) the disposition of the indices of control systems. These require that the structures of these mathematical tools be correctly determined. The structures of the determining matrices have been obtained in Ukwu [2014g and 2014h] for the relevant systems; the construction of the optimal control for transfer of points can be easily achieved using the solution matrices developed in Ukwu and Garba [2014e]. This paper has made a positive contribution by filling in the gap in (iii) through the determination of the structure of the computable indices of the control system (1). This can be effectively deployed for controllability investigation, as reflected in the ensuing examples.

Example 1

Use the control index matrices to determine the Euclidean controllability of the system

$\dot{x}(t)=A_0x(t)+A_1x(t-h)+Bu$, where $A_0=\begin{pmatrix}2, & 0\\ 0, & 2\end{pmatrix}, A_1=\begin{pmatrix}0, & 1\\ 0, & 0\end{pmatrix}, B=\begin{pmatrix}1\\ -2\end{pmatrix}.$

Solution

From theorem 2 and remark 2,

$$X(\tau,t_1)=\begin{cases} e^{a(t_1-\tau)}I_n,\ \tau\in K_0; \\ e^{a(t_1-\tau)}I_n+\sum_{i=1}^{j} A_1^i \dfrac{(t_1-\tau-ih)^i}{i!} e^{a(t_1-\tau-ih)},\ \tau\in K_j. \end{cases}$$

$$a=2, p=2 \Rightarrow X(\tau,t_1)=\begin{cases} e^{2(t_1-\tau)}I_n,\ \tau\in K_0; \\ \begin{pmatrix}1 & (t_1-\tau-h)e^{-2h}\\ 0 & 0\end{pmatrix} e^{2(t_1-\tau)},\ \tau\in K_j,\ j\geq 1 \end{cases}$$

$\tau\in K_0, c^*X(\tau,t_1)B=0\Rightarrow (c_1,c_2)\begin{pmatrix}1\\ -2\end{pmatrix}=0\Rightarrow c_1=2c_2\Rightarrow c=\alpha\begin{pmatrix}2\\ 1\end{pmatrix}, \alpha$ arbitrary $\Rightarrow$

the given system is not Euclidean controllable on $J_0 = [0, h]$, for any positive delay h.

$$j \ge 1, \tau \in K_j, c^* X(\tau, t_1) B = 0 \Rightarrow (c_1, c_2) \begin{pmatrix} 1, & (t_1 - \tau - h) e^{-2h} \\ 0, & 0 \end{pmatrix} e^{2(t_1 - \tau)} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = 0$$

$$\Rightarrow (c_1, c_2) \begin{pmatrix} 1 - 2(t_1 - \tau - h) e^{-2h} \\ 0 \end{pmatrix} = 0 \Rightarrow (1 - 2(t_1 - \tau - h) e^{-2h}) c_1 \ne 0, \; c_2 \text{ arbitrary.}$$

We conclude that the system is not Euclidean controllable on $[kh, (k + 1)h]$ for any positive integral k such that $(k+1)h < t_1$.

Example 2

Use the control index matrices to determine control disposition of the system

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + Bu, \text{ where } A_0 = \begin{pmatrix} -1, & 0 \\ 0, & -1 \end{pmatrix}, A_1 = \begin{pmatrix} 0, & 0 \\ 1, & 0 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$a = -1, p = 2 \Rightarrow X(\tau, t_1) = \begin{cases} e^{-1(t_1 - \tau)} I_n, \; \tau \in K_0; \\ \begin{pmatrix} 1 & 0 \\ (t_1 - \tau - h) e^h & 0 \end{pmatrix} e^{-1(t_1 - \tau)}, \; \tau \in K_j, j \ge 1 \end{cases}$$

$$\tau \in K_0, c^* X(\tau, t_1) B = 0 \Rightarrow (c_1, c_2) \begin{pmatrix} e^{-1(t_1 - \tau)} \\ 0 \end{pmatrix} = 0 \Rightarrow c_1 = 0 \Rightarrow c = \alpha \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \alpha \text{ arbitrary} \Rightarrow$$

the given system is not Euclidean controllable on J_0, for any $h > 0$.

$$j \ge 1, \tau \in K_j, c^* X(\tau, t_1) B = 0 \Rightarrow (c_1, c_2) \begin{pmatrix} 1 & 0 \\ (t_1 - \tau - h) e^h & 0 \end{pmatrix} e^{-1(t_1 - \tau)} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$$

$$\Rightarrow (c_1, c_2) \begin{pmatrix} 1 \\ (t_1 - \tau - h) e^h \end{pmatrix} = 0 \Rightarrow c_1 + c_2 (t_1 - \tau - h) e^h = 0 \Rightarrow c_1 + c_2 (t_1 - h) e^h - c_2 \tau = 0.$$

We invoke the linear independence of 1 and τ to deduce that $c_2 = 0, c_1 = 0 \Rightarrow c = 0 \Rightarrow$ the given system is Euclidean controllable on $[h, t_1), t_1 > h.$ Hence the given system is Euclidean controllable on $[0, t_1]$. However we cannot assert that it is controllable since it is not controllable on the sub-interval $[0, t_1]$ with $t_1 = h > 0$.

CONCLUSION

This article obtained the expressions for the control index matrices of (1) on the finite interval $[t_1 - 4h, t_1]$,with explicit determination of their analytic dispositions; in the sequel it obtained global results on those matrices for various diagonal and nilpotency contingencies of the coefficient matrices in (1), effectively obviating the need to start from the interval $[t_1 - h, t_1]$ in order to compute the control index matrices and solutions for problem instances and then use successively the method of steps to extend these to the intervals $[t_1 - (j+1)h, \; t_1 - jh]$, for positive integral $j : t_1 - (j+1)h \ge 0$. The implications are wide-ranging. By applying the generalized results on the intervals $[t_1 - (j+1)h,\ t_1 - jh]$, the solutions of the corresponding terminal function problems can be more readily obtained. Furthermore appropriate indices of control systems can be constructed and consequently the interrogation of the controllability disposition can be undertaken and the twin issue of the feasibility of admissible controls for transfers of points associated with controllability problems can be settled, based on the non-singularity or otherwise of the Controllability Grammian; needless to say that the appropriate optimal controls can be constructed for a problem instance if the Grammian is invertible.

Finally, the article demonstrated an aspect of its utility by using the indices of control systems to investigate the Euclidean controllability of system (1) for two problem instances.

FUTURE RESEARCH

The results in this article will be extended to single-delay autonomous linear neutral systems as well as double-delay autonomous linear control systems. The latter will exploit skillful combinations of summation notations, the greatest integer function and multiple integrals along with the method of steps and backward continuation recursive procedure.

REFERENCES

Banks, H. T. and Kent G. A.(1972) *Control of functional differential equations of retarded and neutral type to targets in function space,* SIAM J. Control, Vol. 10, Iss. 4.

Jacobs, M. Q. and Langenhop. (1976) *Criteria for Function Space Controllability of Linear Neutral Systems.* SIAM J. Control Optim. Vol. 14, 1009-1048.

Dauer, J. P. and Gahl, R. D. (1977) *Controllability of nonlinear delay systems.* J.O.T.A. Vol. 21, Iss.1.

Angell, T. S. (1980) On Controllability for Nonlinear Hereditary Systems; a fixed point approach. Nonlinear Analysis, Theory, Methods and Applications 4, 529-548.

Onwuatu, J. U. (1984) *On the null-controllability in function space of nonlinear systems of neutral functional differential equations with limited controls.* J. Optimiz. Theory Appl. Vol., 42, 397 – 420.

Balachandran, K.(1986) *Controllability of Nonlinear Systems with Delays in both State and Control.* Kybernetika, Vol. 22, Iss. 4.

Underwood, R. G. and Chukwu, E. N. (1988) *Null Controllability of Nonlinear neutral Differential Equations.* J. Math. Analy. And Appl. Vol., 129, 474-483.

Balachandran, K.(1992) *Controllability of Neutral Volterra Integro-differential Systems.* J. Austral. Math. Soc. Ser. B 34, 18-25

Balachandran, K. and Balasubramaniam, P. (1993) *A Note on Controllability of Neutral Volterra Integro-differential Systems.* Journal of Applied Mathematics and Stochastic Analysis Vol. 6, Iss. 2, Summer, 153-160.

Chukwu, E.N. (2001) Differential Models and Neutral Systems for Controlling the Wealth of Nations. Series on Advances in Mathematics for Applied Sciences (Book 54), World Scientific Pub Co Inc; 1st edition.

Chukwu, E. N. (2002) Stability and Time-Optimal Control of Hereditary Systems with Application to the Economic Dynamics of the US. Series: Stability and Control: Theory, Methods and Applications. CRC Press.

Chukwu, E. N. (2003) Optimal Control of the Growth of Wealth of Nations Series: Stability and Control: Theory, Methods and Applications. CRC Press.

Iheagwam, V. A. (2003) *Characterization of Cores of Targets for Ordinary systems with Distributed Delays in Control.* JNMS, 2003, Vol., 22, 23-34.

Balachandran, K. , and Anandhi, E.. R.(2004) *Controllability of neutral integro-differential infinite delay systems in Banach spaces.* Taiwanese Journal of Mathematics Vol., 8, Iss., 4, 689- 702.

Davis, I., Jackreece, P. (2005) *Controllability and Null Controllability of Linear Systems.* Journal of Applied Sciences and Environment Management. Vol., 9, Iss., 3, 31-36.

Machado, J.A., Ravichandran, C., Rivero, M., and Trujillo, J.J. (2013) Controllability results for impulsive mixed – type functional integro-differential evolution equations with nonlocal conditions. Fixed Point Theory and Applications 2013, 2013:66.

Chang, Y. K., and Chalishajar, D. N.(2008) *Controllability of mixed Volterra-Fredholm type integro- differential inclusions in Banach spaces.* J. Franklin Inst. Vol., 345, 499-507.

Chang , Y. K., Anguraj, A., Mallika, A. M.(2009) *Controllability of impulsive neutral differential inclusions with infinite delay in Banach spaces.* Chaos Solitons Fractals. Vol., 39, Iss. ,4, 1864-1876.

Balachandran, K., Annapoorani, N.(2009) Existence results for impulsive neutral evolution integro-differential equations with infinite delay. Nonlinear Anal.. 3, 674-684.

Ye, R. (2010) Existence of solutions for impulsive partial neutral functional differential equations with infinite delay. Nonlinear Anal.. 73, 155-162.

Vijayakumar, V., Sivasankaran, S. Mallika, A. M. (2011) Existence of global solutions for second order impulsive abstract functional integro- differential equations. Dyn. Contin. Discrete Impuls. Syst.. 18. 747 – 766.

Vijayakumar, V., Sivasankaran, S. Mallika, A. M. (2012) *Existence of global solutions for second order impulsive neutral functional integro- differential equations with infinite delay*. Nonlinear Stud.. Vol., 19, Iss., 2, 327 – 343.

Selvi, S., Mallika, A. M.(2012) *Controllability results for impulsive differential systems with finite delay*. J. Nonlinear Sci. Appl.. Vol. 5, 2006 – 219.

Chukwu, E. N. (1992) Stability and Time-optimal control of hereditary systems: Academic Press, New York.

Gabasov, R. and Kirillova, F. (1976) The qualitative theory of optimal processes: Marcel Dekker Inc., New York.

Hale, J. K. (1977) *Theory of functional differential equations.* Applied Mathematical Science, Vol. 3, Springer-Verlag, New York.

Manitius, A. (1978) Control Theory and Topics in Functional Analysis. Vol. 3, Int. Atomic Energy Agency, Vienna.

Tadmore, G. (1984) *Functional differential equations of retarded and neutral types: Analytical solutions and piecewise continuous controls.* J. Differential equations, Vol. 51, No. 2, Pp. 151-181.

Ukwu, C. (1987) *Compactness of cores of targets for linear delay systems.*, J. Math. Analy. and Appl., Vol. 125, No. 2, August 1. pp. 323-330.

Ukwu, C. (1992) Euclidean Controllability and Cores of Euclidean Targets for Differential difference systems. Master of Science Thesis in Applied Math. with O.R. (Unpublished), North Carolina State University, Raleigh, N. C. U.S.A.

Ukwu, C. (1996) *An exposition on Cores and Controllability of differential difference systems*, ABACUS Vol. 24, No. 2, pp. 276-285.

Ukwu, C. (2014g) *The structure of determining matrices for a class of double delay control systems.* International Journal of Mathematics and Statistics Invention (IJMSI), Vol. 2, Iss., 3, March 2014.

Ukwu, C. (2014h) The structure of determining matrices for single-delay autonomous linear neutral control systems. International Journal of Mathematics and Statistics Invention (IJMSI), Vol.2, Iss., 3, March 2014.

Ukwu, C. and E. J. D. Garba (2014e) *Derivation of an optimal expression for solution matrices of a class of single-delay scalar differential equations*. Journal of Nigerian Association of Mathematical Physics, March, 2014.