



OPTIMAL MACHINE REPLACEMENT STRATEGIES USING DYNAMIC PROGRAMMING RECURSIONS: A CASE STUDY OF NASCO HOUSEHOLD PRODUCTS LTD., JOS

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ABSTRACT

This study investigated the economic value that may accrue to the industrial system due to machine replacement decisions with respect to packing machines at Nasco Household Products Limited, Jos. A questionnaire and direct observation methods were used to obtain data for the study and the data collected were interrogated to determine their reliability. Using the verified pertinent data, the investigation obtained optimal replacement strategies which revealed that the machine should be replaced based on the age transition diagrams from the solution yield of the machine replacement optimization problem. The results were achieved using the established structure of the sets of feasible machine ages corresponding to the various decision periods, in machine replacement problems, which eliminated the need for cumbersome network diagrams for the dynamic recursions. The dynamic programming recursions were found to be inherently manually computationally intensive and so did not lend themselves to sensitivity analyses of the problem parameters.

1. INTRODUCTION

The equipment replacement optimization (ERO) is an important topic in operations research, applied mathematics, statistics, economics, industrial engineering and management science. Items which are under constant usage need replacement at appropriate times, as the efficiency of the operating system that uses such items deteriorates, thus resulting in rising operating and maintenance (O&M) costs and decreasing salvage values Taha [1].

The equipment replacement optimization (ERO) problem has been studied by a lot of researchers (Bellman [2]) and there has been an enormous amount of research on the ERO with finite age horizon using the Deterministic Dynamic Programming (DDP) approach (Hartman & Murphy [3]; Hillier & Liberman [4]). However, it should be noted that almost all the previous research efforts have been devoted to the (DDP) solution formulation and its limited applications to extremely simplified case study/or examples. To the best of our knowledge, there have been no research efforts made so far (except Fan et al. [5][6]) on the application of such DP approaches to solving real-life ERO problems. The DP approach will undoubtedly be the preferred approach to solving the ERO problem because it can explicitly consider the uncertainty in the machine and the annual O&M cost accordingly. Meyer [7] perhaps due to computational constraints, is one among the very few to study the ERO problem technology.

As a case study of machine replacement optimization (ERO) problems, the industrial management needs to determine the optimal revenue that may be accrued in their industry from a packing machine within a ten-year period and to take a decision to either keep or replace the machine. The current investigation included reward functionals that are more helpful in the industry because it uses inputs such as, equipment purchase price, revenue, and salvage value. These functionals are in the form of dynamic programming recursions which are used appropriately to make effective decisions. The study aims at assessing the economic value that an industrial system may accrue due to packing machine replacement, using the Machine Replacement Dynamic Programming (ERDP) model as a decision making tool. This aim shall be achieved by: ascertaining the optimal decision making policy of ‘Keeping or Replacing’ a machine within ten-year period of its lifetime; calculating the maximum net income that is attainable by the industrial system through a ‘Keep or Replace’ decision on packing machines within the ten-year period; determining the most economic age of a packing machine in the industrial system and computing the optimal replacement decisions and associated rewards in Nasco Household Product Ltd. Jos.

2. MATERIAL AND METHODS

The problem data, working definitions, elements of the DP model and the dynamic programming (DP) recursions are laid out as follows:

Equipment Starting age = t_1

Equipment Replacement age = m

S_i = The set of feasible equipment ages (states) in decision period i (say year i), $i \in \{1, 2, \dots, n\}$

$s(t)$ = salvage value of a t – year old equipment; $t = 0, 1, \dots, m$

I = fixed cost of acquiring a new equipment in any year

2.1 The Elements of the DP are the Following:

1. Stage i , represented by year i , $i \in \{1, 2, \dots, n\}$
2. The alternatives at stage (year) i . These call for keeping or replacing the equipment at the beginning of year i
3. The state at stage (year) i , represented by the age of the equipment at the beginning of year i .

Let $f_i(t)$ be the maximum net income for years $i, i+1, \dots, n-1, n$ given that the equipment is t years old at the beginning of year i .

Note: The definition of $f_i(t)$ starting from year i to year n implies that backward recursion will be used. Forward recursion would start from year 1 to year i .

The following theorem is applicable to the backward recursive procedure:

2.2 Theorem 1: Theorem on Dynamic Programming (DP) Recursions

$$f_i(t) = \max \begin{cases} r(t) - c(t) + f_{i+1}(t+1); \text{ IF KEEP} \\ r(0) + s(t) - I - c(0) + f_{i+1}(1); \text{ REPLACE} \end{cases}$$

$$f_{n+1}(x) = s(x), \quad i = 0, 1, \dots, n-1, \quad x = \text{age of machine at the start of period } n+1$$

2.2.1 Proof of Theorem 1

Decision: KEEP

$r(t) - c(t)$ = Net revenue (income) from operating the equipment an extra year when it is already t years old. Then the equipment age advances to $t+1$ years and hence $f_{i+1}(t+1)$ = maximum income for years $i+1, \dots, n$ given that the equipment is $t+1$ years old at the start of year $i+1$.

Decision: REPLACE

$r(0) = 1$ – year revenue from a new equipment (age 0)

$c(0) =$ cost of operating a new equipment for 1 year (from the start of year i to the end of year i)

$I =$ fixed cost of a new equipment

$s(t) =$ salvage cost for a t – year old equipment

$$\text{Net income} = \underbrace{r(0) - c(0)}_{\substack{\text{net income for operating} \\ \text{the new equipment from the} \\ \text{beginning of each feasible year to the} \\ \text{end of that year. The age of the} \\ \text{equipment then advances to 1 year}}} + s(t) - I + \underbrace{f_{i+1}(1)}_{\substack{\text{max net income for} \\ \text{years } i+1, \dots, n \text{ given} \\ \text{that the machine is 1-year} \\ \text{old at the start of year } i+1}}$$

$f_{n+1}(x) = s(x)$ or $f_{n+1}(\cdot) = s(\cdot) \Rightarrow$ Sell off the machine at the end of the planning horizon at price $s(\cdot)$, regardless of its age, with no further income realized from the beginning of year $n+1$, since the planning horizon length is n years. Therefore the recursive equation is correct. This completes the proof.

2.3 Pertinent Remarks on the DP Recursions

For $i \in \{1, 2, \dots, n\}$, $f_i(t)$ may be identified as $f_i(t) = \max_{\{K,R\}} \{f_i^K(t), f_i^R(t)\}$, where

$$f_i^K(t) = r(t) - c(t) + f_{i+1}(t+1) \text{ and } f_i^R(t) = r(0) + s(t) - I - c(0) + f_{i+1}(1)$$

For $i \in \{1, 2, \dots, n\}$ and $t \in S_i$, the optimal decision may be identified as $D_i(t)$, where

$$D_i(t) = \underset{\{K,R\}}{\operatorname{argmax}} g_i(t, K, R); g_i(t, K, R) = \begin{cases} f_i^K(t), & \text{if Decision is KEEP} \\ f_i^R(t), & \text{if Decision is REPLACE} \end{cases}$$

Define

$$x_i = \begin{cases} 1, & \text{if decision is REPLACE in stage } i \text{ (start of decision year } i) \\ 0, & \text{if decision is KEEP in stage } i \text{ (start of decision year } i) \end{cases}$$

Then

$$g_i(t, K, R) = f_i(t) = (1-x_i)f_i^K(t) + x_i f_i^R(t), i \in \{1, 2, \dots, n\}$$

If the revenue profile is not given then $r(t)$ may be set identically equal to zero, in which case

$-f_i(t) =$ minimum cost associated with operating the equipment from the start of decision year i to the end of decision year n .

If the variable cost profile is not given then $c(t)$ may be set identically equal to zero, in which case

$f_i(t) =$ maximum net revenue from the start of decision year i to the end of decision year n .

If the cost profile is not given then $c(t)$ and I may be set identically equal to zero, in which case

$f_i(t)$ = maximum accrueable revenue from the start of decision year i to the end of decision year n .

If the equipment is bought new at the beginning of the decision year 1, then the optimal return is

$g_1(0) = -I + f_1(0)$ or $-g_1(0) = I - f_1(0)$, as appropriate.

2.4 Theorem 2: Theorem on Analytic Determination of the Set of Feasible Ages at Each Stage Ukwu [8]

Let S_i denote the set of feasible equipment ages at the start of the decision year i . Let t_i denote the age of the machine at the start of the decision year i , that is, $S_i = \{t_i\}$. Then for $i \in \{1, 2, \dots, n\}$,

$$S_i = \begin{cases} \left\{ \min_{2 \leq j \leq i} \{j-1, m\} \right\} \cup \{1 + (i-2+t_i) \operatorname{sgn}(\max\{m+2-t_i-i, 0\})\}, & \text{if } t_i < m \\ \left\{ \min_{2 \leq j \leq i} \{j-1, m\} \right\}, & \text{if } t_i \geq m \end{cases}$$

2.4.1 Corollary 1

If $t_i < m$, then for $i \in \{2, \dots, n\}$,

$$S_i = \begin{cases} \left\{ \min_{2 \leq j \leq i} \{j-1, m\} \right\} \cup \{i-1+t_i\}, & \text{if } i \leq m+1-t_i \\ \left\{ \min_{2 \leq j \leq i} \{j-1, m\} \right\}, & \text{if } i > m+1-t_i \end{cases}$$

2.4.2 Corollary 2

$$S_i = \{t_i\}. \text{ If } m \leq n \text{ and } t_i \in \{0, 1\}, \text{ then for } i \in \{2, \dots, n\}, S_i = \left\{ \min_{2 \leq j \leq i+t_i} \{j-1, m\} \right\}$$

This result is a corollary to Theorem 1, in [8].

2.4.3 Corollary 3

If $t_i < m$ and $m > n$, then for $i \in \{2, \dots, n\}$,

$$S_i = \begin{cases} \{1, \dots, i-1, i-1+t_i\}, & \text{if } i \leq m+1-t_i \\ \{1, \dots, i-1\}, & \text{if } i > m+1-t_i \end{cases}$$

2.4.4 Corollary 4

If the mandatory replacement age restriction is waived, then

$$S_i = \{t_i\}, S_i = \{1, 2, \dots, i-1, i-1+t_i\}, i \in \{2, \dots, n\}.$$

2.5 Imperativeness of Theorem 2

The sketches in Figures 1 and 2, represent the network diagrammatic insight for chosen $\{m, n, t_1\}$ data set that necessitated the need for theorem 2.

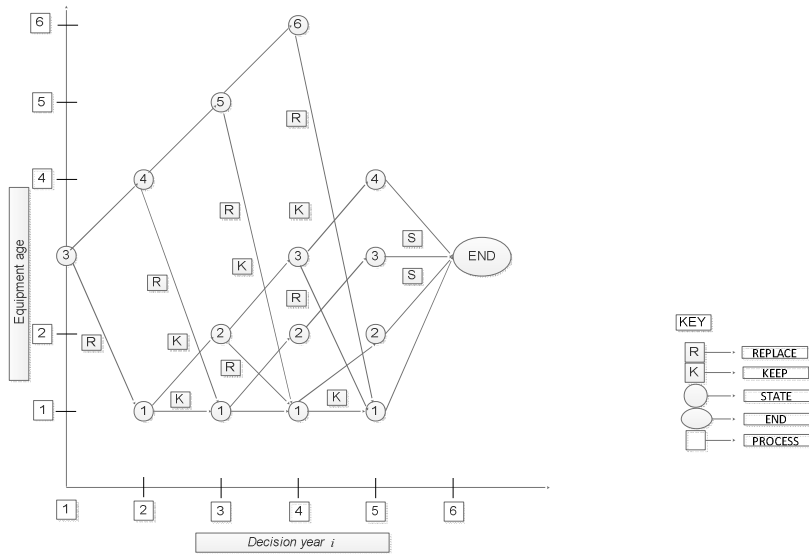


Figure 1: Network Diagram for Equipment Replacement Problem with Starting Age 3, Mandatory Replacement Age 6 for a 4-year Planning Horizon.

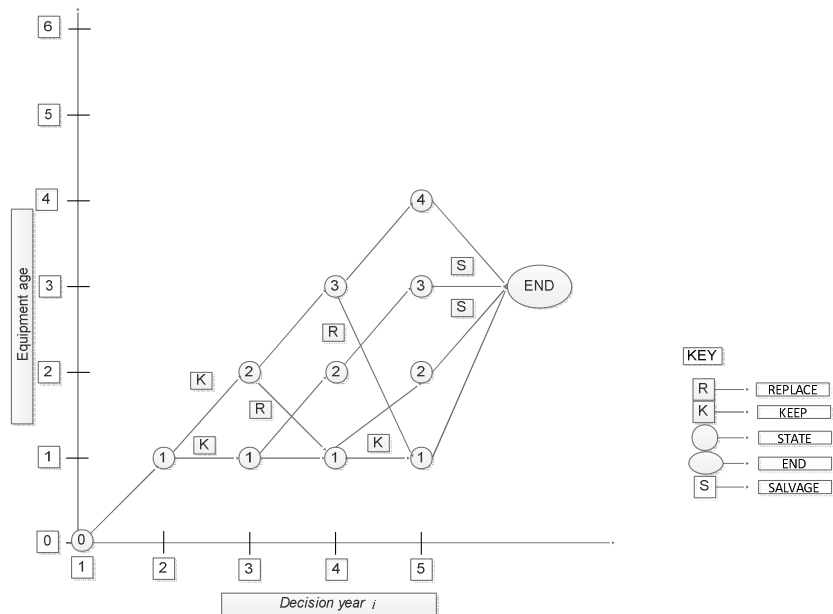


Figure 2: Network Diagram for Equipment Replacement Problem with Starting Age 0, Mandatory Replacement Age 6, for 4-year Planning Horizon.

Clearly, such network diagrams are intractable for large-horizon problems, demonstrating the imperativeness of theorem 2.

3. RESULTS AND DISCUSSION

3.1 Case Study Application and the Implementation of the Solution Templates

NASCO HOUSEHOLD PRODUCTS LTD. JOS needs to determine the optimal replacement policy for a Pakona (pk70 packing) machine over the next ten years. The following table gives the pertinent data for the problem. The company does not prescribe any mandatory replacement age. The cost of a new Pakona (pk70 packing) machine is ₦8,608,000.

TABLE 1: Pertinent Data for Optimal Policy and Reward Determination

Age: t yrs.	Revenue: $r(t)$ (₦)	Operating cost: $c(t)$	Salvage value: $s(t)$
0	2,330,000	240,000	-
1	2,320,000	253,000	8,177,600
2	2,210,000	257,000	7,768,720
3	2,090,000	272,000	7,380,284
4	1,895,000	274,000	7,011,269
5	1,770,000	301,000	6,310,142
6	1,720,000	311,000	5,679,127
7	1,655,000	361,000	5,111,215
8	1,590,000	396,000	4,600,093
9	1,345,000	403,000	3,910,079
10	1,029,000	415,000	3,323,567

Age: t years; Revenue: $r(t)$; Operating cost: $c(t)$; Salvage value: $s(t)$

Set $v(0) = r(0) - c(0) - I$, in Theorem 1, so that the 'REPLACE' component of $f_i(t)$ becomes $v(0) + s(t) + f_{i+1}(1) = -6,518,000 + s(t) + f_{i+1}(1)$.

The above problem is solved for $t_1 \in \{0,1\}$, using Corollary 5 and the dynamic programming recursions in Theorem 1, since there is no prescription of mandatory replacement age. The results are summarized in the tables below with the starting age of 0:

STARTING AGE 0

Table 2: Stage 10 Computational Results

Age t (yrs.)	0	1	2	3	4	5
Revenue: $r(t)$	2330000	2320000	2210000	2090000	1895000	1770000
Mnt. cost, $c(t)$	240000	253000	257000	272000	274000	301000
Salvage value,		8177600	7768720	7380284	7011269	6310142

$s(t)$						
<i>K</i>		9835720	9333284	8829269	7931142	7148127
<i>R</i>		9837200	9428320	9039884	8670869	7969742
Opt. value: $f(t)$		9837200	9428320	9039884	8670869	7969742
Opt. Decision, D^*		<i>R</i>	<i>R</i>	<i>R</i>	<i>R</i>	<i>R</i>
State		1	2	3	4	5

Age t (yrs.)	6	7	8	9	10
Revenue: $r(t)$	1720000	1655000	1590000	1345000	1029000
Mnt. cost, $c(t)$	311000	361000	396000	403000	415000
Salvage value, $s(t)$					
<i>K</i>	5679127	5111215	4600093	3910079	3323567
<i>R</i>	6520215	5894093	5104079	4265567	
<i>R</i>	7338727	6770815	6259693	5569679	
Opt. value: $f(t)$	7338727	6770815	6259693	5569679	
D^*	<i>R</i>	<i>R</i>	<i>R</i>	<i>R</i>	
State	6	7	8	9	

Table 3: Stage 9 Computational Results

Age t (yrs.)	1	2	3	4
<i>K</i>	9835720	9333284	8829269	7931142
<i>R</i>	9837200	9428320	9039884	8670869
Opt. value: $f(t)$	9837200	9428320	9039884	8670869
D^*	<i>R</i>	<i>R</i>	<i>R</i>	<i>R</i>
State	1	2	3	4

Stage 9 Computational Results Contd.

Age t (yrs.)	5	6	7	8
<i>K</i>	7148127	6520215	5894093	5104079
<i>R</i>	7969742	7338727	6770815	6259693
Opt. value: $f(t)$	7969742	7338727	6770815	6259693
D^*	<i>R</i>	<i>R</i>	<i>R</i>	<i>R</i>
State	5	6	7	8

Table 4: Stage 8 Computational Results

Age t	1	2	3	4	5	6	7
<i>K</i>	1315492 0	12652484	12148469	11250342	10467327	9839415	9213293
<i>R</i>	1315640 0	12747520	12359084	11990069	11288942	1065792 7	10090015

$f(t)$	1315640 0	12747520	12359084	11990069	11288942	1065792 7	10090015
D^*	R	R	R	R	R	R	R
State	1	2	3	4	5	6	7

Table 5: Stage 7 Computational Results

Age t	1	2	3	4	5	6
K	14814520	14312084	13808069	12909942	12126927	11499015
R	14816000	14407120	14018684	13649669	12948542	12317527
$f(t)$	14816000	14407120	14018684	13649669	12948542	12317527
D^*	R	R	R	R	R	R
State	1	2	3	4	5	6

Table 6: Stage 6 Computational Results

Age t	1	2	3	4	5
K	16474120	15971684	15467669	14569542	13786527
R	16475600	16066720	15678284	15309269	14608142
$f(t)$	16475600	16066720	15678284	15309269	14608142
D^*	R	R	R	R	R
State	1	2	3	4	5

Table 7: Stage 5 Computational Results

Age t	1	2	3	4
K	18133720	17631284	17127269	16229142
R	18135200	17726320	17337884	16968869
$f(t)$	18135200	17726320	17337884	16968869
D^*	R	R	R	R
State	1	2	3	4

Table 8: Stage 4 Computational Results Stage 3 Computational Results

Age t	1	2	3		1	2
K	19793320	19290884	18786869		21452920	20950484
R	19794800	19385920	18997484		21454400	21045520
$f(t)$	19794800	19385920	18997484		21454400	21045520
D^*	R	R	R		R	R
State	1	2	3		1	2

Table 9: Stage 2 Results Stage 1 Results

Age t	1	0
K	23112520	25204000
R	23114000	16596000
$f(t)$	23114000	25204000
D^*	R	K
State	1	0

3.1.1 Age transition diagram for the determination of the optimal policy prescriptions and the corresponding returns:

$$0K1R1R1R1R1R1R1R1R1S$$

$$f_1(0) = (\text{The maximum net income from year 0 to 10}) = \text{₹ } 25,204,000$$

Note that in general, there are $2(n+1)$ concatenated age and decision symbols corresponding to the horizon length n .

The above diagrams are consistent with $n=10$

Interpretation

Start with a new Pakona (pk70 packing) machine; keep the machine for the next one year, and then replace it, resulting in a 1-year machine at the beginning of the succeeding year. Sell off the last replacement machine at the end of the 10-year period.

STARTING AGE 1

Table 10: Stage 10 Computational Results

Age t (yrs.)	0	1	2	3	4	5
Revenue: $r(t)$	2330000	2320000	2210000	2090000	1895000	1770000
Mnt. cost, $c(t)$	240000	253000	257000	272000	274000	301000
Salvage value, $s(t)$		8177600	7768720	7380284	7011269	6310142
K	9835720	9333284	8829269	7931142	7148127	9835720
R	9837200	9428320	9039884	8670869	7969742	9837200
Opt. value: $f(t)$	9837200	9428320	9039884	8670869	7969742	9837200
Opt. Decision	R	R	R	R	R	R
State	1	2	3	4	5	1

Stage 10 Computational Results Contd.

Age t (yrs.)	6	7	8	9	10
Revenue: $r(t)$	1720000	1655000	1590000	1345000	1029000
Mnt. cost, $c(t)$	311000	361000	396000	403000	415000
Salvage value, $s(t)$	5679127	5111215	4600093	3910079	3323567
K	6520215	5894093	5104079	4265567	614000
R	7338727	6770815	6259693	5569679	4983167
Opt. value: $f(t)$	7338727	6770815	6259693	5569679	4983167
Opt. Decision	R	R	R	R	R
State	6	7	8	9	10

Table 11: Stage 9 Computational Results

Age t (yrs.)	1	2	3	4
K	9835720	9333284	8829269	7931142
R	9837200	9428320	9039884	8670869
Opt. value: $f(t)$	9837200	9428320	9039884	8670869
Opt. Decision	R	R	R	R
State	1	2	3	4

Stage 9 Computational Results Contd.

Age t (yrs.)	5	6	7	8	9
K	8807727	8179815	7553693	6763679	5925167
R	9629342	8998327	8430415	7919293	7229279
Opt. value: $f(t)$	9629342	8998327	8430415	7919293	7229279
Opt. Decision	R	R	R	R	R
State	5	6	7	8	9

Table 12: Stage 8 Computational Results

T	1	2	3	4	5	6	7	8
K	1315492 0	1265248 4	1214846 9	1125034 2	1046732 7	9839415	9213293	842327 9
R	1315640 0	1274752 0	1235908 4	1199006 9	1128894 2	1065792 7	1009001 5	957889 3
$f(t)$	1315640 0	1274752 0	1235908 4	1199006 9	1128894 2	1065792 7	1009001 5	957889 3
D^*	R	R	R	R	R	R	R	R
$s(t)$	1	2	3	4	5	6	7	8

Table 13: Stage 7 Computational Results

Age t	1	2	3	4	5	6	7
K	14814520	14312084	13808069	12909942	12126927	11499015	10872893
R	14816000	14407120	14018684	13649669	12948542	12317527	11749615
$f(t)$	14816000	14407120	14018684	13649669	12948542	12317527	11749615
D^*	R	R	R	R	R	R	R
State	1	2	3	4	5	6	7

Table 14: Stage 6 Computational Results

Age t	1	2	3	4	5	6
K	16474120	15971684	15467669	14569542	13786527	13158615
R	16475600	16066720	15678284	15309269	14608142	13977127
$f(t)$	16475600	16066720	15678284	15309269	14608142	13977127
D^*	R	R	R	R	R	R
State	1	2	3	4	5	6

Table 15: Stage 5 Computational Results

Age t	1	2	3	4	5
K	18133720	17631284	17127269	16229142	15446127
R	18135200	17726320	17337884	16968869	16267742
$f(t)$	18135200	17726320	17337884	16968869	16267742
D^*	R	R	R	R	R
State	1	2	3	4	5

Table 16: Stage 4 Computational Results Stage 3 Computational Results

Age t	1	2	3	4		1	2	3
K	19793320	19290884	18786869	17888742		21452920	20950484	20446469
R	19794800	19385920	18997484	18628469		21454400	21045520	20657084
$f(t)$	19794800	19385920	18997484	18628469		21454400	21045520	20657084
D^*	R	R	R	R		R	R	R
State	1	2	3	4		1	2	3

Stage 2 Results

Stage 1 Results

Age t	1	2		1
K	23112520	22610084		24772120
R	23114000	22705120		24773600
$f(t)$	23114000	22705120		24773600
D^*	R	R		R
State	1	2		1

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