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THE STRUCTURE OF TRANSITION MATRICES FOR CERTAIN SINGLE DELAY SYSTEMS AND APPLICATION INSTANCES

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ABSTRACT
This paper investigated the expressions and structure of the transition matrices for single - delay autonomous linear systems on the interval [0, 4*h*]. Sequel to this, the paper obtained the structure of the transition matrices when any of the system's coefficient matrices is diagonal. The development of the transition matrices exploited the continuity of these matrices for positive time periods and the method of steps to obtain these matrices on successive intervals of length equal to the delay *h*. The paper also discussed the analytic disposition of these matrices as well as provided elucidating application instances to the solutions of initial function problems.

1. INTRODUCTION

The qualitative approach to the controllability of functional differential control systems has witnessed a tremendous surge of interests and activities for the past fifty years among control theorists and applied mathematicians in general. This circumvents the severe difficulties associated with the search for and computations of solutions of such systems.

Unfortunately computations of solutions cannot be wished away in the tracking of trajectories and many practical applications. Literature on state space approach to control studies is replete with variation of constants formulas, which incorporate the solution matrices of the free part of the systems. See Chukwu (1992), Gabsov and Kirillova (1976), Hale (1977), Manitius (1978), Tadmore (1984), and Ukwu (1987, 1992,1996). Regrettably no author has made any attempt to obtain general expressions for such solution matrices involving the delay h. The usual approach is to start from the interval $[0, h]$ and compute the solution matrices and solutions for given problem instances and then use the method of steps to extend these to the intervals $[kh, (k+1)h]$, for positive integral k, not exceeding 2, for the most part. Such approach is rather restrictive and doomed to failure in terms of structure for arbitrary k. In other words such approach fails to address the issue of the structure and computing complexity of solution matrices and solutions quite vital for real-world applications. The need to address such short-comings has become imperative; this is the major contribution of this paper, with its wide-ranging implications for extensions to more general systems and holistic approach to controllability studies.

We consider the class of single-delay differential systems

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h),\ t \in \mathbf{R} \quad (1)$$

where A_0 and A_1 are $n \times n$ constant matrices.

Let $Y(t)$ be a generic solution matrix of (1) for any $t \in \mathbf{R}$, where

$$Y(t) = \begin{cases} I_n,\ t = 0, \\ 0,\ t < 0. \end{cases}$$

and I_n is the identity matrix of order n. Let $J_k = [kh,(k+1)h], k \in \{0,1,\cdots\}$.

By the definition of solution matrices,

$$\dot{Y}(t) = A_0 Y(t) + A_1 Y(t-h) \ a.e, \ t \in \mathbf{R}$$

Hence

$$e^{-A_0 t}\left[\dot{Y}(t) - A_0 Y(t)\right] = \frac{d}{dt}\left[e^{-A_0}Y(t)\right] = A_1 Y(t-h) \ a.e$$

The transition matrices will be obtained piece – wise on successive intervals of length h.

2. THEORETICAL ANALYSIS

2.1 Theorem: Transition Matrix Formula for Autonomous, Single - Delay Linear Differential Equations (1) on $[0, 4h]$

Let $Y(t)$ be the transition matrix of (1). Then

$$Y(t) = \begin{cases} e^{A_0 t}, \ t \in J_0; & (2) \\ e^{A_0 t} + \int_h^t e^{A_0(t-s)} A_1 e^{A_0(s-h)} ds, \ t \in J_1; & (3) \\ e^{A_0 t} + \int_h^t e^{A_0(t-s)} A_1 e^{A_0(s-h)} ds + \int_{2h}^t \int_h^{s_2-h} e^{A_0(t-s_2)} A_1 e^{A_0(s_2-s_1-h)} A_1 e^{A_0(s_1-h)} ds_1 ds_2, t \in J_2; & (4) \\ e^{A_0 t} + \int_h^t e^{A_0(t-s)} A_1 e^{A_0(s-h)} ds + \int_{2h}^t \int_h^{s_2-h} e^{A_0(t-s_2)} A_1 e^{A_0(s_2-s_1-h)} A_1 e^{A_0(s_1-h)} ds_1 ds_2 & \\ \quad + \int_{3h}^t \int_{2h}^{s_3-h} \int_h^{s_2-h} e^{A_0(t-s_3)} A_1 e^{A_0(s_3-h-s_2)} A_1 e^{A_0(s_2-h-s_1)} A_1 e^{A_0(s_1-h)} ds_1 ds_2 ds_3, t \in J_3 & (5) \end{cases}$$

Proof

On $(0,h)$, $Y(t-h) = 0 \Rightarrow \dot{Y}(t) = A_0 Y(t)$ a.e. on $[0,h] \Rightarrow Y(t) \equiv Y_1(t) = e^{A_0 t}C; Y(0) = I_n \Rightarrow C = I_n$

$$\Rightarrow Y(t) = e^{At} \text{ on } J_0 \qquad (6)$$

Consider the interval J_1. Then on $(h, 2h)$,

$$t-h \in (0,h) \Rightarrow \dot{Y}(t) = A_0 Y(t) + A_1 e^{A_0(t-h)} \Rightarrow \frac{d}{dt}\left[e^{-A_0 t}Y(t)\right] = e^{-A_0 t}\left[\dot{Y}(t) - A_0 Y(t)\right] = A_1 e^{A_0(t-h)}$$

$$\Rightarrow \frac{d}{dt}\left[e^{-A_0 t}Y(t)\right] = e^{-A_0 t}\left[\dot{Y}(t) - A_0 Y(t)\right] = e^{-A_0 t}A_1 e^{A_0(t-h)} \Rightarrow e^{-A_0 t}Y(t) - e^{-A_0 h}Y(h) = \int_h^t e^{-A_0 s}A_1 e^{A_0(s-h)} ds$$

$$\Rightarrow Y(t) = e^{A_0(t-h)}Y(h) + \int_h^t e^{A_0(t-s)} A_1 e^{A_0(s-h)} ds, \text{ on } J_1.$$

$$Y(h) = e^{A_0 h} \Rightarrow Y(t) = e^{A_0 t} + \int_h^t e^{A_0(t-s)} A_1 e^{A_0(s-h)} ds, \text{ on } J_1 \qquad (7)$$

Consider the interval J_2; then $t \in J_2 \Rightarrow t-h \in J_1 \Rightarrow s_2 - h \in J_1$.

$$t \in (2h,\,3h) \Rightarrow \dot{Y}(t) - A_0 Y(t) = A_1 Y(t-h) \Rightarrow e^{A_0 t}\frac{d}{dt}\left[e^{-A_0 t}Y(t)\right] = A_1 Y(t-h)$$

$$\Rightarrow Y(t) = e^{A_0(t-2h)}Y(2h) + \int_{2h}^{t} e^{A_0(t-s_2)}A_1 Y(s_2-h)ds_2$$

$$\Rightarrow Y(t) = e^{A_0(t-2h)}Y(2h) + \int_{2h}^{t} e^{A_0(t-s_2)}\left[A_1 e^{A_0(s_2-h)} + \int_{h}^{s_2-h} e^{A_0(s_2-s_1-h)}A_1 e^{A_0(s_1-h)}ds_1\right]ds_2, \text{ on } J_2$$

By the continuity of $Y(t)$, $Y(2h) = e^{2hA_0} + \int_{h}^{2h} e^{A_0(2h-s_1)}A_1 e^{A_0(s_1-h)}ds_1$. Therefore,

$$Y(t) = e^{A_0 t} + \int_{h}^{2h} e^{A_0(t-s_1)}A_1 e^{A_0(s_1-h)}ds_1 + \int_{2h}^{t} e^{A_0(t-s_2)}A_1 e^{A_0(s_2-h)}ds_2 + \int_{2h}^{t}\int_{h}^{s_2-h} e^{A_0(t-s_2)}A_1 e^{A_0(s_2-s_1-h)}A_1 e^{A_0(s_1-h)}ds_1 ds_2, \text{ on } J_2$$

$$\Rightarrow Y(t) = e^{A_0 t} + \int_{h}^{t} e^{A_0(t-s)}A_1 e^{A_0(s-h)}ds + \int_{2h}^{t}\int_{h}^{s_2-h} e^{A_0(t-s_2)}A_1 e^{A_0(s_2-s_1-h)}A_1 e^{A_0(s_1-h)}ds_1 ds_2, \text{ on } J_2 \quad (8)$$

Clearly, on J_3,

$$e^{-A_0 t}Y(t) - e^{-3A_0 h}Y(3h) = \int_{3h}^{t} e^{-A_0 s_3}A_1 Y(s_3-h)ds_3$$

$$= \int_{3h}^{t} e^{-A_0 s_3}A_1\left[e^{A_0(s_3-h)} + \int_{h}^{2h} e^{A_0(s_3-h-s_1)}A_1 e^{A_0(s_1-h)}ds_1 + \int_{2h}^{s_3-h} e^{A_0(s_3-h-s_2)}A_1 e^{A_0(s_2-h)}ds_2\right]ds_3$$
$$+ \int_{3h}^{t}\int_{2h}^{s_3-h}\int_{h}^{s_2-h} e^{-A_0 s_3}A_1 e^{A_0(s_3-h-s_2)}A_1 e^{A_0(s_2-h-s_1)}A_1 e^{A_0(s_1-h)}ds_1 ds_2 ds_3$$

$$\Rightarrow Y(t) = \int_{3h}^{t} e^{A_0(t-s_3)}A_1\left[e^{A_0(s_3-h)} + \int_{h}^{2h} e^{A_0(s_3-h-s_1)}A_1 e^{A_0(s_1-h)}ds_1 + \int_{2h}^{s_3-h} e^{A_0(s_3-h-s_2)}A_1 e^{A_0(s_2-h)}ds_2\right]ds_3$$
$$+ \int_{3h}^{t}\int_{2h}^{s_3-h}\int_{h}^{s_2-h} e^{A_0(t-s_3)}A_1 e^{A_0(s_3-h-s_2)}A_1 e^{A_0(s_2-h-s_1)}A_1 e^{A_0(s_1-h)}ds_1 ds_2 ds_3$$
$$+ \int_{h}^{2h} e^{A_0(t-s_1)}A_1 e^{A_0(s_1-h)}ds_1 + \int_{2h}^{3h} e^{A_0(t-s_2)}A_1 e^{A_0(s_2-h)}ds_2 + \int_{2h}^{3h}\int_{h}^{s_2-h} e^{A_0(t-s_2)}A_1 e^{A_0(s_2-h-s_1)}A_1 e^{A_0(s_1-h)}ds_1 ds_2$$

$$\Rightarrow Y(t) = e^{A_0 t} + \int_{3h}^{t} e^{A_0(t-s_3)}A_1 e^{A_0(s_3-h)}ds_3 + \int_{3h}^{t}\int_{h}^{2h} e^{A_0(t-s_3)}A_1 e^{A_0(s_3-h-s_1)}A_1 e^{A_0(s_1-h)}ds_1 ds_3$$
$$+ \int_{3h}^{t}\int_{2h}^{s_3-h} e^{A_0(t-s_3)}A_1 e^{A_0(s_3-h-s_2)}A_1 e^{A_0(s_2-h)}ds_2 ds_3$$
$$+ \int_{3h}^{t}\int_{2h}^{s_3-h}\int_{h}^{s_2-h} e^{A_0(t-s_3)}A_1 e^{A_0(s_3-h-s_2)}A_1 e^{A_0(s_2-h-s_1)}A_1 e^{A_0(s_1-h)}ds_1 ds_2 ds_3$$
$$+ \int_{h}^{2h} e^{A_0(t-s_1)}A_1 e^{A_0(s_1-h)}ds_1 + \int_{2h}^{3h} e^{A_0(t-s_2)}A_1 e^{A_0(s_2-h)}ds_2 + \int_{2h}^{3h}\int_{h}^{s_2-h} e^{A_0(t-s_2)}A_1 e^{A_0(s_2-h-s_1)}A_1 e^{A_0(s_1-h)}ds_1 ds_2$$

$$\Rightarrow Y(t) = e^{A_0 t} + \int_h^t e^{A_0(t-s_1)} A_1 e^{A_0(s_1-h)} ds_1 + \int_{2h}^t \int_h^{s_2-h} e^{A_0(t-s_2)} A_1 e^{A_0(s_2-h-s_1)} A_1 e^{A_0(s_1-h)} ds_1 ds_2$$

$$+ \int_{3h}^t \int_{2h}^{s_3-h} \int_h^{s_2-h} e^{A_0(t-s_3)} A_1 e^{A_0(s_3-h-s_2)} A_1 e^{A_0(s_2-h-s_1)} A_1 e^{A_0(s_1-h)} ds_1 ds_2 ds_3, \text{on } J_3 \quad (9)$$

This completes the proof.

2.2 Investigation of Analyticity

$Y(t) = 0 \ \forall t < 0 \Rightarrow \lim_{t\to 0^-} Y(t) = 0, \lim_{t\to 0^-} \dot{Y}(t) = 0; Y(t) = e^{A_0 t} \ \forall\, t \in [0,h] \Rightarrow \lim_{t\to 0^+} Y(t) = I, \ \lim_{t\to 0^+} \dot{Y}(t) = A_0.$

Therefore $Y(t)$ is not continuous at $t = 0$ and certainly not differentiable there. It follows that $Y(t)$ is not analytic at $t = 0$.

$\lim_{t\to h^-} \dot{Y}(t) = A_0 e^{A_0 h}; \ \lim_{t\to h^+} \dot{Y}(t) = A_0 \lim_{t\to h^+} Y(t) + A_1 \lim_{t\to h^+} Y(t-h) = A_0 Y(h) + A_1 Y(0) = A_0 e^{A_0 h} + A_1,$

by the continuity of $Y(t)$ for $t > 0$ and the appropriate evaluations using the interval J_0 for the left limit and J_1 for the right limit. Alternatively, we apply Leibniz's rule for differentiating an integral to $\lim_{t\to h^+} Y(t)$ to obtain $A_0 e^{A_0(2h-h)} + A_1 e^{A_0(h-h)} - 0 + \int_h^h A_0 e^{A_0(t-s)} A_1 e^{A_0(t-s)} ds = A_0 e^{A_0 h} + A_1.$

Therefore $Y(t)$ is not differentiable at $t = h$ and hence not analytic there.

$$\lim_{t\to(2h)^-} \dot{Y}(t) = A_0 \lim_{t\to(2h)^-} Y(t) + A_1 \lim_{t\to(2h)^-} Y(t-h) = A_0 Y(2h) + A_1 Y(h), \text{by the continuity of } Y(t), \forall t > 0$$

$$\Rightarrow \lim_{t\to(2h)^-} Y(t) = A_0 e^{2A_0 h} + \int_h^{2h} A_0 e^{A_0(2h-s)} A_1 e^{A_0(s-h)} ds + A_1 e^{A_0 h}$$

$$= A_0 e^{2A_0 h} + A_1 e^{A_0 h} + \int_h^{2h} A_0 e^{A_0(2h-s)} A_1 e^{A_0(s-h)} ds$$

$$\lim_{t\to(2h)^+} \dot{Y}(t) = A_0 \lim_{t\to(2h)^+} Y(t) + A_1 \lim_{t\to(2h)^+} Y(t-h) = A_0 Y(2h) + A_1 Y(h), \text{by the continuity of } Y(t), \forall t > 0$$

$$\Rightarrow \lim_{t\to(2h)^+} \dot{Y}(t) = A_0 Y(2h) + A_1 Y(h) = A_0 \left[e^{2A_0 h} + \int_h^{2h} e^{A_0(2h-s_1)} A_1 e^{A_0(s_1-h)} ds_1 \right]$$

$$+ A_1 \left[e^{A_0 h} + \int_h^h e^{A_0(h-s_1)} A_1 e^{A_0(s_1-h)} ds_1 \right]$$

$$= A_0 e^{2A_0 h} + A_1 e^{A_0 h} + \int_h^{2h} e^{A_0(2h-s_1)} A_1 e^{A_0(s_1-h)} ds_1.$$

The same result is obtainable using Leibniz's rule, verifying the consistency of our results and the differentiability of that solution matrix at the indicated point.

Let us examine the left and right limits of the second derivative at $t = 2h$.

$$t \in (2h, 3h) \Rightarrow \lim_{t\to(2h)^+} \ddot{Y}(t) = A_0 \lim_{t\to(2h)^+} \dot{Y}(t) + A_1 \lim_{t\to(2h)^+} \dot{Y}(t-h)$$

$$= A_0^2 e^{2A_0 h} + A_0 A_1 e^{A_0 h} + A_0 \int_h^{2h} e^{A_0(2h-s_1)} A_1 e^{A_0(s_1-h)} ds_1 + A_1 A_0 e^{A_0 h} + A_1^2$$

$$= \left[A_0A_1 + A_1A_0\right]e^{A_0h} + A_0^2e^{2A_0h} + A_1^2 + A_0\int_h^{2h} e^{A_0(2h-s_1)}A_1e^{A_0(s_1-h)}ds_1$$

$$t \in (2h, 3h) \Rightarrow \lim_{t\to(2h)^-}\ddot{Y}(t) = A_0\lim_{t\to(2h)^-}\dot{Y}(t) + A_1\lim_{t\to(2h)^-}\dot{Y}(t-h)$$

$$= A_0\left[A_0e^{2A_0h} + A_1e^{A_0h} + \int_h^{2h} e^{A_0(2h-s_1)}A_1e^{A_0(s_1-h)}ds_1\right] + A_1A_0e^{A_0h}$$

$$= \left[A_0A_1 + A_1A_0\right]e^{A_0h} + A_0^2e^{2A_0h} + A_0\int_h^{2h} e^{A_0(2h-s_1)}A_1e^{A_0(s_1-h)}ds_1 \Rightarrow \lim_{t\to(2h)^+}\ddot{Y}(t) = \lim_{t\to(2h)^-}\ddot{Y}(t) + A_1^2$$

$\Rightarrow \ddot{Y}(t)$ does not exist at $t = 2h$ and hence $Y(t)$ is not analytic there.

Straight – forward application of Leibniz's rule yields

$$\lim_{t\to(3h)+}\dot{Y}(t) = \left[A_0e^{A_0t} + A_1e^{A_0(t-h)} + A_0\int_{3h}^{t} e^{A_0(t-s_3)}A_1e^{A_0(s_3-h)}ds_3 + \int_h^{2h} A_1e^{A_0(s_3-h-s_1)}A_1e^{A_0(s_1-h)}ds_1\right]_{t=(3h)^+}$$

$$+\left[A_0\int_{3h}^{t}\int_h^{2h} e^{A_0(t-s_3)}A_1e^{A_0(s_3-h-s_1)}A_1e^{A_0(s_1-h)}ds_1ds_3\right]_{t=(3h)^+} + 0 + 0$$

$$+ A_0\left[\int_h^{2h} e^{A_0(3h-s_1)}A_1e^{A_0(s_1-h)}ds_1 + \int_{2h}^{3h} e^{A_0(3h-s_2)}A_1e^{A_0(s_2-h)}ds_2 + \int_{2h}^{3h}\int_h^{s_2-h} e^{A_0(3h-s_2)}A_1e^{A_0(s_2-h-s_1)}A_1e^{A_0(s_1-h)}ds_1ds_2\right]$$

$$\Rightarrow \lim_{t\to(3h)^+}\dot{Y}(t) = \left[A_0e^{3A_0h} + A_1e^{2A_0h} + \int_h^{2h} A_1e^{A_0(2h-s_1)}A_1e^{A_0(s_1-h)}ds_1\right]$$

$$+ A_0\left[\int_h^{3h} e^{A_0(3h-s_1)}A_1e^{A_0(s_1-h)}ds_1 + \int_{2h}^{3h}\int_h^{s_2-h} e^{A_0(3h-s_2)}A_1e^{A_0(s_2-h-s_1)}A_1e^{A_0(s_1-h)}ds_1ds_2\right]$$

$$t = (3h)^- \Rightarrow t \in J_2 \Rightarrow \lim_{t\to(3h)^-}\dot{Y}(t) = A_0\left[e^{3A_0h} + \int_h^{3h} e^{A_0(3h-s_1)}A_1e^{A_0(s_1-h)}ds_1\right] + A_1e^{2A_0h}$$

$$+\int_h^{2h} A_1e^{A_0(2h-s_1)}A_1e^{A_0(s_1-h)}ds_1 + A_0\int_{2h}^{3h}\int_h^{s_2-h} e^{A_0(3h-s_2)}A_1e^{A_0(s_2-s_1-h)}A_1e^{A_0(s_1-h)}ds_1ds_2$$

$\Rightarrow \lim_{t\to(3h)^+}\dot{Y}(t) \neq \lim_{t\to(3h)^-}\dot{Y}(t) \Rightarrow Y(t)$ is not differentiable at $t = 3h$, and hence not analytic there.

We conclude that $Y(t)$ is analytic on the interval $[0,\ 4h)$ except on the set $\{0,1,2,3\}$.
See Chidume (2007) for discussions on analytical functions.

3. RESULTS AND ANALYSIS

3.1 Corollary

If $A_0 = 0,$ then

$$Y(t) = \begin{cases} I_n,\ t \in J_0; \\ I_n + A_1(t-h),\ t \in J_1; \\ I_n + A_1(t-h) + A_1^2 \dfrac{(t-2h)^2}{2!}, t \in J_2; \\ I_n + A_1(t-h) + A_1^2 \dfrac{(t-2h)^2}{2!} + A_1^3 \dfrac{(t-3h)^3}{3!},\ t \in J_3 \end{cases}$$

3.2 Corollary

If $A_1 = \text{diag}(b),$ then

$$Y(t) = \begin{cases} e^{A_0 t},\ t \in J_0; \\ e^{A_0 t} + b(t-h)e^{A_0(t-h)},\ t \in J_1; \\ e^{A_0 t} + b(t-h)e^{A_0(t-h)} + b^2 \dfrac{(t-2h)^2}{2!} e^{A_0(t-2h)}, t \in J_2; \\ e^{A_0 t} + b(t-h)e^{A_0(t-h)} + b^2 \dfrac{(t-2h)^2}{2!} e^{A_0(t-2h)} + b^3 \dfrac{(t-3h)^3}{3!} e^{A_0(t-3h)},\ t \in J_3 \end{cases}$$

3.3 Theorem: Generalization of Theorem 2.1 for $A_0 = \text{diag}(a).$

Consider the system (1) with $A_0 = \text{diag}(a).$ Then the transition matrices are given by

$$Y(t) = \begin{cases} e^{at} I_n,\ t \in J_0; \\ e^{at} I_n + \displaystyle\sum_{i=1}^{k} A_1^i \frac{(t-ih)^i}{i!} e^{a(t-ih)},\ t \in J_k. \end{cases}$$

Proof

The theorem is valid for $k \in \{0,1,2,3\}$, as seen from the last corollary. Assume the validity of the theorem for $t \in J_p, 4 \le p \le k,$ for some $k \ge 5.$ Then on J_{k+1},

$$Y(t) = e^{A_0(t-[k+1]h)} Y([k+1]h) + \int_{[k+1]h}^{t} e^{A_0(t-s_{k+1})} A_1 Y(s_{k+1}-h) ds_{k+1}$$

$$= e^{a(t-[k+1]h)} e^{a[k+1]h} I_n + e^{a(t-[k+1]h)} \sum_{i=1}^{k} A_1^i \frac{([k+1-i]h)^i}{i!} e^{a[k+1-i]h}$$

$$+ \int_{[k+1]h}^{t} e^{a(t-s_{k+1})} A_1 \left[I_n e^{a(s_{k+1}-h)} + \sum_{i=1}^{k} A_1^i \frac{(s_{k+1}-[i+1]h)^i}{i!} e^{a(s_{k+1}-[i+1]h} \right] ds_{k+1}$$

$$= I_n e^{at} + \sum_{i=1}^{k} A_1^i \frac{([k+1-i]h)^i}{i!} e^{a(t-ih)} + A_1(t-[k+1]h) e^{a(t-h)}$$

$$+ \sum_{i=1}^{k} A_1^{i+1} \left[\frac{(s_{k+1}-[i+1]h)^{i+1}}{(i+1)!} e^{a(t-[i+1]h)} \right]_{[k+1]h}^{t}$$

$$= I_n e^{at} + \sum_{i=1}^{k} A_1^i \frac{([k+1-i]h)^i}{i!} e^{a(t-ih)} + A_1(t-[k+1]h)e^{a(t-h)} + \sum_{i=1}^{k} A_1^{i+1} \frac{(t-[i+1]h)^{i+1}}{(i+1)!} e^{a(t-[i+1]h)}$$
$$- \sum_{i=1}^{k} A_1^{i+1} \frac{([k-i]h)^{i+1}}{(i+1)!} e^{a([k-i]h)}$$

$$= I_n e^{at} + \sum_{i=1}^{k} A_1^i \frac{([k+1-i]h)^i}{i!} e^{a(t-ih)} + A_1(t-[k+1]h)e^{a(t-h)} + \sum_{i=2}^{k+1} A_1^{i} \frac{(t-ih)^{i}}{i!} e^{a(t-ih)}$$
$$- \sum_{i=2}^{k+1} A_1^{i} \frac{([k+1-i]h)^{i}}{i!} e^{([k+1-i]h)}$$

$$= I_n e^{at} + A_1(kh)e^{a(t-h)} + A_1(t-[k+1]h)e^{a(t-h)} + \sum_{i=2}^{k+1} A_1^{i} \frac{(t-ih)^{i}}{i!} e^{a(t-ih)}$$

$$= I_n e^{at} + A_1(t-h)e^{a(t-h)} + \sum_{i=2}^{k+1} A_1^{i} \frac{(t-ih)^{i}}{i!} = I_n e^{at} + \sum_{i=1}^{k+1} A_1^{i} \frac{(t-ih)^{i}}{i!} e^{a(t-ih)}, t \in J_{k+1}$$

completing the proof of the theorem.

Remark 1: The special case $A_0 = 0$.

$$A_0 = 0 \Rightarrow Y(t) = \begin{cases} I_n, t \in J_0; \\ I_n + \sum_{i=1}^{k} A_1^i \frac{(t-ih)^i}{i!}, \; t \in J_k, k \geq 1. \end{cases}$$

Remark 2: The special case $A_0 = 0$ and A_1 is nilpotent of index $p < k$, then

$$Y(t) = \begin{cases} I_n, t \in J_0; \\ I_n + \sum_{i=1}^{p-1} A_1^i \frac{(t-ih)^i}{i!}, \; t \in J_k, k \geq 1. \end{cases}$$

3.4 Theorem: Generalization of Theorem 2.1 for $A_1 = \text{diag}(b)$.

If $A_1 = \text{diag}(b)$, then

$$Y(t) = \begin{cases} e^{A_0 t}, \; t \in J_0; \\ e^{A_0 t} + \sum_{i=1}^{k} b^i \frac{(t-ih)^i}{i!} e^{A_0(t-ih)}, \; t \in J_k, \; k \geq 1 \end{cases}$$

Proof

The theorem is valid for $k \in \{0,1,2,3\}$, as seen from the last corollary. Assume the validity of the theorem for $t \in J_p, 4 \leq p \leq k,$ for some $k \geq 5.$ Then on J_{k+1},

$$Y(t) = e^{A_0(t-[k+1]h)} Y([k+1]h) + \int_{[k+1]h}^{t} e^{A_0(t-s_{k+1})} A_1 Y(s_{k+1}-h) ds_{k+1}$$

$$\Rightarrow Y(t) = e^{A_0(t-[k+1]h)} \left(e^{A_0[k+1]h} + \sum_{i=1}^{k} b^i \frac{([k+1-i]h)^i}{i!} e^{A_0([k+1-i]h)} \right)$$
$$+ \int_{[k+1]h}^{t} e^{A_0(t-s_{k+1})} b \left[e^{A_0(s_{k+1}-h)} + \sum_{i=1}^{k} b^i \frac{(s_{k+1}-[i+1]h)^i}{i!} e^{A_0(s_{k+1}-[i+1]h)} \right] ds_{k+1}$$

$$\Rightarrow Y(t) = e^{A_0 t} + \sum_{i=1}^{k} b^i \frac{([k+1-i]h)^i}{i!} e^{A_0(t-ih)}$$
$$+ b(t-[k+1]h)e^{A_0(t-h)} + \sum_{i=1}^{k} b^{i+1} \int_{[k+1]h}^{t} \frac{(s_{k+1}-[i+1]h)^i}{i!} e^{A_0(t-[i+1]h)} ds_{k+1}$$
$$\Rightarrow Y(t) = e^{A_0 t} + \sum_{i=1}^{k} b^i \frac{([k+1-i]h)^i}{i!} e^{A_0(t-ih)}$$
$$b(t-[k+1]h)e^{A_0(t-h)} + \left[\sum_{i=1}^{k} b^{i+1} \frac{(s_{k+1}-[i+1]h)^{i+1}}{(i+1)!} e^{A_0(t-[i+1]h)} \right]_{[k+1]h}^{t}$$
$$= e^{A_0 t} + \sum_{i=1}^{k} b^i \frac{([k+1-i]h)^i}{i!} e^{A_0(t-ih)} + b(t-[k+1]h)e^{A_0(t-h)}$$
$$+ \sum_{i=1}^{k} b^{i+1} \frac{(t-[i+1]h)^{i+1}}{(i+1)!} e^{A_0(t-[i+1]h)} - \sum_{i=1}^{k} b^{i+1} \frac{([k-i]h)^{i+1}}{(i+1)!} e^{A_0([k-i]h)}$$
$$= e^{A_0 t} + \sum_{i=1}^{k} b^i \frac{([k+1-i]h)^i}{i!} e^{A_0(t-ih)} + b(t-[k+1]h)e^{A_0(t-h)}$$
$$+ \sum_{i=2}^{k+1} b^i \frac{(t-ih)^i}{i!} e^{A_0(t-ih)} - \sum_{i=2}^{k+1} b^i \frac{([k+1-i]h)^i}{i!} e^{A_0([k+1-i]h)}$$
$$= e^{A_0 t} + b(kh)e^{A_0(t-h)} + b(t-[k+1]h)e^{A_0(t-h)} + \sum_{i=2}^{k+1} b^i \frac{(t-ih)^i}{i!} e^{A_0(t-ih)}$$
$$= e^{A_0 t} + b(t-h)e^{A_0(t-h)} + \sum_{i=2}^{k+1} b^i \frac{(t-ih)^i}{i!} e^{A_0(t-ih)} = e^{A_0 t} + \sum_{i=1}^{k+1} b^i \frac{(t-ih)^i}{i!} e^{A_0(t-ih)},$$

proving the theorem.

3.5 Application to Variation of Constants formula

Consider the single – delay system

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + Bu, \text{ where } A_0 = \begin{pmatrix} 0, & 0 \\ 0, & 0 \end{pmatrix}, A_1 = \begin{pmatrix} 0, & 1 \\ 0, & 0 \end{pmatrix}, B = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Obtain the explicit unique solution to the free part (uncontrolled part) of the problem on the interval $[0,\infty)$, given the initial function $\phi(\mathrm{t}) = \begin{pmatrix} t+1 \\ 1 \end{pmatrix}$, on $[-1,0]$.

Solution

The unique solution to the free part of the initial function problem is given by

$$x(t,\phi) = Y(t)\phi(0) + \int_{-h}^{0} Y(t-s-h)A_1\phi(s)ds, \; t \ge 0$$

There is no direct straight-forward application of the above formula in one fell swoop– a fact that is hardly emphasized by control practitioners. The method of steps must be applied by reasoning as follows:

$s \in [-h,0] \Rightarrow -s-h \in [-h,0]; t \in [0,h] \Rightarrow t-s-h \in [t-h,t] \subset [-h,h]; t-s-h > 0$ iff $s < t-h.$

From (1) and Remark 1, we obtain the following.

$$\text{On } J_0,\ Y(t-s-h)=\begin{cases}0, & s>t-h\\ I_n, & s\le t-h\end{cases}$$

Noting that $t-s-h>0 \Rightarrow t-s-h\in(0,h]$, we conclude that on the t – interval [0, h], with $h=1$,

$$x(t,\phi,0)=\begin{pmatrix}1\\1\end{pmatrix}+\int_{-1}^{t-1}\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}1+s\\1\end{pmatrix}ds$$

$$x(t,\phi,0)=\begin{pmatrix}1\\1\end{pmatrix}+\int_{-1}^{t-1}\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}ds=\begin{pmatrix}1\\1\end{pmatrix}+\begin{pmatrix}t\\0\end{pmatrix}=\begin{pmatrix}1+t\\1\end{pmatrix},\ t\in J_0.$$

$t\in[1,2]\Rightarrow t-s-h\in[0,2]; t-s-h\ge 1$ iff $s\le t-2$, i.e., $s\in[-1,t-2]; t-s-h\le 1$ iff $s\in[t-2,0]$.

$$Y(t-s-1)\ =\ \begin{cases}I_n, & 0\le t-s-1\le 1\\ I_n+A_1(t-s-2), & 1\le t-s-1\le 2\end{cases}$$

Hence

$$x(t,\phi,0)=\left[I_n+A_1(t-1)\right]\phi(0)+\int_{-1}^{t-2}\left[I_n+A_1(t-s-2)\right]A_1\phi(s)ds+\int_{t-2}^{0}I_nA_1\phi(s)ds,\ t\in J_1$$

$$=\left[\begin{pmatrix}1&0\\0&1\end{pmatrix}+\begin{pmatrix}0&t-1\\0&0\end{pmatrix}\right]\begin{pmatrix}1\\1\end{pmatrix}+\int_{-1}^{t-2}\left[\begin{pmatrix}1&0\\0&1\end{pmatrix}+\begin{pmatrix}0&t-s-2\\0&0\end{pmatrix}\right]\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}1+s\\1\end{pmatrix}ds+\int_{t-2}^{0}\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}1+s\\1\end{pmatrix}ds$$

$$=\begin{pmatrix}t\\1\end{pmatrix}+\int_{-1}^{t-2}\begin{pmatrix}1\\0\end{pmatrix}ds+\int_{t-2}^{0}\begin{pmatrix}1\\0\end{pmatrix}ds=\begin{pmatrix}t\\1\end{pmatrix}+\int_{-1}^{0}\begin{pmatrix}1\\0\end{pmatrix}ds=\begin{pmatrix}t+1\\1\end{pmatrix}$$ Indeed, $x(t,\phi,0)=\begin{pmatrix}t+1\\1\end{pmatrix}, \forall t\ge -1,$ since A_1

is nilpotent of index 2. To see this note that

$$Y(t)=I_n+A_1(t-1)=\begin{pmatrix}1&t-1\\0&1\end{pmatrix},\ \forall t\in J_k, k\ge 2;\ Y(t)\phi(0)=\begin{pmatrix}t\\1\end{pmatrix},\ t-s-1\in[k-1,k].$$

Hence $t-s-1>0\Rightarrow x(t,\phi,0)=\begin{pmatrix}t\\1\end{pmatrix}+\int_{-1}^{0}\begin{pmatrix}1&t-s-1\\0&1\end{pmatrix}\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}1+s\\1\end{pmatrix}ds=\begin{pmatrix}t+1\\1\end{pmatrix}.$

If $A_1=\operatorname{diag}(b)$, then $x(t,\phi,0)=\begin{pmatrix}1\\1\end{pmatrix}+\int_{-1}^{t-1}b\begin{pmatrix}s+1\\1\end{pmatrix}ds=\begin{pmatrix}1+\frac{bt^2}{2}\\1+bt\end{pmatrix},\ t\in J_0.$

$$x(t,\phi,0)=\left[\begin{pmatrix}1&0\\0&1\end{pmatrix}+\begin{pmatrix}0&t-1\\0&0\end{pmatrix}\right]\begin{pmatrix}1\\1\end{pmatrix}+\int_{-1}^{t-2}\left[\begin{pmatrix}1&0\\0&1\end{pmatrix}+\begin{pmatrix}0&t-s-2\\0&0\end{pmatrix}\right]b\begin{pmatrix}1+s\\1\end{pmatrix}ds+\int_{t-2}^{0}b\begin{pmatrix}1+s\\1\end{pmatrix}ds$$

$$x(t,\phi,0)=\begin{pmatrix}t\\1\end{pmatrix}+\int_{-1}^{t-2}\left[\begin{pmatrix}1&0\\0&1\end{pmatrix}+\begin{pmatrix}0&t-s-2\\0&0\end{pmatrix}\right]b\begin{pmatrix}1+s\\1\end{pmatrix}ds+\int_{t-2}^{0}b\begin{pmatrix}1+s\\1\end{pmatrix}ds,\ t\in J_1$$

$$= \begin{pmatrix} t \\ 1 \end{pmatrix} + b \begin{pmatrix} \frac{t^2}{2} - t + 1 \\ 1 \end{pmatrix} = \begin{pmatrix} b\frac{t^2}{2} + [1-b]t + b \\ b+1 \end{pmatrix}, t \in J_1$$

$$t \in J_k, k \geq 2 \Rightarrow Y(t) = I_n + \sum_{i=1}^{k} \frac{b^i}{i!} I_n (t-i)^i = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sum_{i=1}^{k} \frac{b^i}{i!} \begin{pmatrix} [t-i]^i & 0 \\ 0 & [t-i]^i \end{pmatrix}.$$

$$\Rightarrow t-s-1 \in J_{k-1} \Rightarrow Y(t-s-1) = I_n + \sum_{i=1}^{k-1} \frac{b^i}{i!} I_n (t-s-1-i)^i$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sum_{i=1}^{k-1} \frac{b^i}{i!} \begin{pmatrix} [t-s-1-i]^i & 0 \\ 0 & [t-s-1-i]^i \end{pmatrix}; \; Y(t)\phi(0) = \sum_{i=1}^{k} \begin{pmatrix} \frac{b^i}{i!}[t-i]^i + 1 \\ \frac{b^i}{i!}[t-i]^i + 1 \end{pmatrix}.$$

Therefore,

$$x(t,\phi,0) = \sum_{i=1}^{k} \begin{pmatrix} \frac{b^i}{i!}[t-i]^i + 1 \\ \frac{b^i}{i!}[t-i]^i + 1 \end{pmatrix} + \int_{-1}^{0} \left[\begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} + \sum_{i=1}^{k-1} \frac{b^{i+1}}{i!} \begin{pmatrix} [t-s-1-i]^i & 0 \\ 0 & [t-s-1-i]^i \end{pmatrix} \right] \begin{pmatrix} s+1 \\ 1 \end{pmatrix} ds, t \in J_k$$

$$= \sum_{i=1}^{k} \begin{pmatrix} \frac{b^i}{i!}[t-i]^i + 1 \\ \frac{b^i}{i!}[t-i]^i + 1 \end{pmatrix} + \int_{-1}^{0} \left[b \begin{pmatrix} s+1 \\ 1 \end{pmatrix} + \sum_{i=1}^{k-1} \frac{b^{i+1}}{i!} \begin{pmatrix} [t-s-1-i]^i (s+1) \\ [t-s-1-i]^i \end{pmatrix} \right] ds, t \in J_k, k \geq 2$$

$$= \sum_{i=1}^{k} \begin{pmatrix} \frac{b^i}{i!}[t-i]^i + 1 \\ \frac{b^i}{i!}[t-i]^i + 1 \end{pmatrix} + b \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} + \int_{-1}^{0} \left[+ \sum_{i=1}^{k-1} \frac{b^{i+1}}{i!} \begin{pmatrix} [t-s-1-i]^i (s+1) \\ [t-s-1-i]^i \end{pmatrix} \right] ds, t \in J_k, k \geq 2$$

We can use integration by parts to obtain

$$\sum_{i=1}^{k-1} \int_{-1}^{0} \frac{b^{i+1}}{i!} [t-s-1-i]^i (s+1) ds = -\sum_{i=1}^{k-1} \frac{(t-1-i)^{i+1}}{(i+1)} - \sum_{i=1}^{k-1} \frac{b^{i+1}}{(i+2)!} \left[(t-1-i)^{i+2} - (t-i)^{i+2} \right], \; k \geq 2$$

$$\Rightarrow x(t,\phi,0) = \left(\frac{k(k+1)}{2} + \sum_{i=1}^{k} \frac{b^i}{i!}[t-i]^i \right) \begin{pmatrix} 1 \\ 1 \end{pmatrix} + b \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$$

$$- \begin{pmatrix} \sum_{i=1}^{k-1} \frac{b^{i+1}}{(i+1)!} (t-1-i)^{i+1} + \sum_{i=1}^{k-1} \frac{b^{i+1}}{(i+2)!} \left[(t-1-i)^{i+2} - (t-i)^{i+2} \right] \\ \sum_{i=1}^{k-1} \frac{b^{i+1}}{(i+1)!} \left[(t-1-i)^{i+1} - (t-i)^{i+1} \right] \end{pmatrix}, \; t \in J_k, k \geq 2$$

CONCLUSION

This article obtained the expressions for the transition matrices of (1) on the finite interval $[0,4h]$,with explicit determination of their analytic dispositions; in the sequel it obtained global results on the transition matrices of (1) for various diagonal and nilpotency contingencies of the coefficient matrices in (1), effectively obviating the need to start from the interval $[0,h]$ in order to compute the transition matrices and solutions for problem instances and then use successively the method of steps to extend these to the intervals $[kh,(k+1)h]$, for positive integral k. The implications are wide-ranging. By applying the generalized results on the intervals $[kh,(k+1)h], k \in \{0,1,\cdots\}$, the solutions of the corresponding initial function problems can be more readily obtained. Furthermore appropriate controllability Grammians can be constructed and consequently the issue of the feasibility of admissible controls for transfers of points associated with controllability problems can be settled, based on the non-singularity or otherwise of the Controllability Grammian; needless to say that the appropriate optimal control can be constructed for a problem instance if the Grammian is invertible.

Finally, the article demonstrated an aspect of its utility by solving an initial function problem on the interval $[0,\infty)$. The solution process revealed that the transition matrices, which are key components in the variation of constants formula, cannot be rightly applied without the correct determination of their J_k domains; glossing over this fact would render the resulting solution deeply flawed.

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