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A Family of Implicit Uniformly Accurate Order Block Integrators for the Solution of $y'' = f(x, y, y')$

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ABSTRACT: *We consider a family of four, five and six-step block methods for the numerical integration of ordinary differential equations of the type $y'' = f(x, y, y')$. The main methods and their additional equations are obtained from the same continuous formulation via interpolation and collocation procedures. The methods which are all implicit are of uniform order and are assembled into a single block equations. These equations which are self starting are simultaneously applied to provide for $y_1, y_2, \dots, y_k$ at once without recourse to any Predictors for the Ordinary differential equations. The order of accuracy, convergence analysis and absolute stability regions of the block methods are also presented.*

KEYWORDS: *Second Order ODE, Continuous formulation, Collocation and Interpolation, Second Order Equations, Block Method.*

I. INTRODUCTION

The study of second order differential equations of the form:

$$y'' = f(x, y, y'),\ y(0) = \alpha \quad y'(0) = \beta \qquad (1)$$

where f is a continuous function, has a huge bibliography covering several applicative fields, from chemistry to physics and engineering. Even if any high order ODE may be recast as a first order one, this transformation increases the size of the original problem and should make its numerical solution more complicated since it requires the computation of both solution and derivatives (which have different slopes) at the same time.

Considerable attention has been devoted to the development of various methods for solving (1) directly without first reducing it to a system of first order differential equations. For instance, Twizell and Khaliq [1], Yusuph and Onumanyi [2], Simos [3], Fatunla [4, 5], Henrici [6], and Lambert [7, 8, 9]. Hairer and Wanner [10] proposed Nystrom type methods and stated order conditions for determining the parameters of the methods. Other methods of the Runge-Kutta type are due to Chawla and Sharma [11]. Methods of the LMM type have been considered by Vigo-Aguiar and Ramos [12, 13] and Awoyemi [14, 15]. In [12], variable stepsize multistep schemes based on the Falkner method were developed and directly applied to (1) in a predictor corrector (PC) mode. In [14, 15] the LMMs were proposed and also implemented in a predictor – corrector mode using the Taylor series algorithm to supply the starting values. Although, the implementation of the methods in a PC mode yielded good accuracy, the procedure is more costly to implement. PC subroutines are very complicated to write for supplying the starting values which lead to longer computer time and more human effort. Our method is cheaper to implement, since it is self-starting and therefore does not share these drawbacks.

In this paper, we develop a family of uniform orders 3, 4 and 5 methods which are applied each as a block to yield approximate solutions $y_1, y_2, \dots, y_k$. We also show that the block methods derived are zero-stable and consistent, hence they methods are convergent.

II. THE DERIVATION OF THE METHOD

In this section, we use the interpolation and collocation procedures to characterize the LMM that is of interest to us by choosing the right number of interpolation points (r) and the right number of collocation points (s). The process leads to a system of equations involving ($r + s$) unknown coefficients, which are determined by the matrix inversion approach. The formula is much easier to derive using the matrix inversion approach (see [2]) rather than using the purely algebraic approach. It is worth noting that LMMs have been widely used to provide the numerical solution of first order systems of IVPs. In this paper, we propose a LMM that is applied directly to (1) without first reducing it to a system of first order ODEs. Although the proposed LMM can be obtained as a finite difference method with constant coefficients as in the conventional fashion, it has more advantages when initially derived and expressed with continuous coefficients for the direct solution of (1). Thus, we approximate the exact solution $y(x)$ by seeking the continuous method $y(x)$of the form:

$$\bar{y}(x) = \sum_{j=0}^{r-1} \alpha_j (x)\, y_{n+j} + h^2 \sum_{j=0}^{s-1} \beta_j(x)\, f_{n+j} \tag{2}$$

where $x \in [a, b]$ and the following notations are introduced. The positive integer $k \geq 2$ denotes the step number of the method (2), which is applied directly to provide the solution to (1). In this light, we seek a solution on

$\pi_N : a = x_0 < x_1 < . . . < x_n < x_{n+1} < \ldots < x_N = b$, $h = x_{n+1} - x_n$, $n = 0,1, \ldots , N$

where π_N is a partition of $[a, b]$ and h is the constant step-size of the partition of π_N. The number of interpolation points r and the number of distinct collocation points s are chosen to satisfy $2 \leq r \leq k$, and $0 < s \leq k + 1$ respectively. We then construct a k-step multistep collocation method of the form (2) by imposing the following conditions:

$$y(x_{n+j}) = y_{n+j},\ j= 0,1, \ldots , r - 1 \,, \tag{3}$$

$$y''(x_{n+j}) = f_{n+j},\ j= 0,1, \ldots , s - 1 \,. \tag{4}$$

Equations (3) and (4) lead to a system of ($r + s$) equations and ($r + s$) unknown coefficients to be determined.

In order to solve this system, we require that the linear k-step method (2) be defined by the assumed polynomial basis functions:

$$\alpha_j (x) = \sum_{j=0}^{r-1} \alpha_{i+1,j} P_i(x) ; j = \{0,1, \ldots , r - 1\} \tag{5}$$

and

$$\beta_j(x) = h^2 \sum_{j=0}^{s-1} \beta_{i+1,j}\, P_i(x); j = \{0,1, \ldots , s - 1\} \tag{6}$$

To obtained $\alpha_j (x)$ and $\beta_j(x)$, [19] arrived at a matrix equation of the form:

$$\mathrm{MH} = \mathrm{I} \tag{7}$$

which imply

$$M = H^{-1} \tag{8}$$

where the constants $\alpha_{i+1,j}$ and $h^2\beta_{i+1,j}$, $j = \{0,1,\dots,r+s-1\}$ are undetermined elements of the $(r+s) \times (r+s)$ matrix M, given by

$$M = \begin{pmatrix} \alpha_{1,0} & \alpha_{1,1} & \cdots & \alpha_{1,r-1} & h^2\beta_{1,0} & \cdots & h^2\beta_{1,s-1} \\ \alpha_{2,0} & \alpha_{2,1} & \cdots & \alpha_{2,r-1} & h^2\beta_{2,0} & \cdots & h^2\beta_{2,s-1} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \alpha_{r+s,0} & \alpha_{r+s,1} & \cdots & \alpha_{r+s,r-1} & h^2\beta_{r+s,0} & \cdots & h^2\beta_{r+s,s-1} \end{pmatrix}$$

(9)

We also define the interpolation/collocation matrix H as

$$H = \begin{pmatrix} P_0(x_n) & \cdots & P_{r+s-1}(x_n) \\ P_0(x_{n+1}) & \cdots & P_{r+s-1}(x_{n+1}) \\ \vdots & & \vdots \\ P_0(x_{n+r-1}) & \cdots & P_{r+s-1}(x_{n+r-1}) \\ P_0''(x_n) & \cdots & P_{r+s-1}''(x_n) \\ P_0''(x_{n+1}) & \cdots & P_{r+s-1}''(x_{n+1}) \\ \vdots & & \vdots \\ P_0''(x_{n+s-1}) & \cdots & P_{r+s-1}''(x_{n+s-1}) \end{pmatrix}$$

(10)

we consider further notations by defining the following vectors:

$$\varphi = (y_n, y_{n+1}, \dots, y_{n+r-1}, f_n, f_{n+1}, \dots, f_{n+s-1})^T$$

$$\Psi(x) = (P_0(x),\ P_1(x),\ \dots,\ P_{r+s-1}(x))^T$$

where T denotes the transpose of the vectors.

The collocation points are selected from the extended set Φ, where

$$\Phi = \{x_n, \ldots, x_{n+k-1}\} \cup \{x_{n+k-1}, x_{n+k}\}$$

III. DERIVATION OF BLOCK METHODS

Case I: FOSBM

Consider the following specifications: k= 4, $r = 4$ *and* $s = 1$ *where* $\{x_n, x_{n+1}, x_{n+2}, x_{n+3}\}$ are interpolation points and $\{x_{n+4}\}$ as collocation point, then following (3) to (10), we obtained H as:

$$\begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 \\ 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 \\ 1 & x_{n+3} & x_{n+3}^2 & x_{n+3}^3 & x_{n+3}^4 \\ 0 & 0 & 2 & 6x_{n+4} & 12\, x_{n+4}^2 \end{pmatrix}$$

(11)

After some manipulations to the matrix inverse of (11), we obtained the continuous formulation

[illegible]

(12)

Evaluating (12) at x_{n+j}, $j = 4$ and its second derivative evaluated at x_{n+j}, $j = 2$ *and* 3, while its 1st derivative is evaluated at x_{n+j}, $j = 0$ yields the following set of discrete equations whose coefficients are reported on table 1.

Case II: FISBM

When the following specifications: k= 5, $r = 5$ *and* $s = 1$ *where* $\{x_n, x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}\}$ are interpolation points and $\{x_{n+5}\}$ as collocation point are considered, then following (3) to (10), we obtained H as:

$$\begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 \\ 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 \\ 1 & x_{n+3} & x_{n+3}^2 & x_{n+3}^3 & x_{n+3}^4 & x_{n+3}^5 \\ 1 & x_{n+4} & x_{n+4}^2 & x_{n+4}^3 & x_{n+4}^4 & x_{n+4}^5 \\ 0 & 0 & 2 & 6x_{n+5} & 12\,x_{n+5}^2 & 20\,x_{n+5}^3 \end{pmatrix}$$

(14)

We do same as in case I, to obtain the continuous formulation of (14) as:

[redacted]

(15)

Evaluating (15) at x_{n+j}, $j = 5$ and its second derivative evaluated at $j = 2, \ldots, 4$, while its 1st derivative is evaluated at $j = 0$ yields the following set of discrete equations, see table 2 for coefficients of the method.

Case III: SISBM

We now look at the following specifications: k= 6, $r = 6$ *and* $s = 1$ *where* $\{x_n, x_{n+1}, x_{n+2}, x_{n+3}, x_{n+4}, x_{n+5}\}$ are interpolation points and $\{x_{n+6}\}$ as collocation point, then H becomes:

$$\begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 \\ 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & x_{n+2}^6 \\ 1 & x_{n+3} & x_{n+3}^2 & x_{n+3}^3 & x_{n+3}^4 & x_{n+3}^5 & x_{n+3}^6 \\ 1 & x_{n+4} & x_{n+4}^2 & x_{n+4}^3 & x_{n+4}^4 & x_{n+4}^5 & x_{n+4}^6 \\ 1 & x_{n+5} & x_{n+5}^2 & x_{n+5}^3 & x_{n+5}^4 & x_{n+5}^5 & x_{n+5}^6 \\ 0 & 0 & 2 & 6x_{n+6} & 12\,x_{n+6}^2 & 20\,x_{n+6}^3 & 20\,x_{n+6}^4 \end{pmatrix}$$

(17)

The continuous formulation of (17) is:

$$y(x) := \left(\frac{-\frac{58997}{24360}\xi}{h} + \frac{14235}{6496}\frac{\xi^2}{h^2} - \frac{37733}{38976}\frac{\xi^3}{h^3} + \frac{2899}{12992}\frac{\xi^4}{h^4} - \frac{4999}{194880}\frac{\xi^5}{h^5} + 1 + \frac{15}{12992}\frac{\xi^6}{h^6} \right) y_n + \left(\frac{\frac{2355}{406}\xi}{h} - \frac{11209}{9744}\frac{\xi^4}{h^4} + \frac{43451}{9744}\frac{\xi^3}{h^3} - \frac{13389}{1624}\frac{\xi^2}{h^2} - \frac{65}{9744}\frac{\xi^6}{h^6} + \frac{1381}{9744}\frac{\xi^5}{h^5} \right) y_{n+1} + \left(\frac{-\frac{5595}{812}\xi}{h} + \frac{307}{19488}\frac{\xi^6}{h^6} + \frac{42981}{3248}\frac{\xi^2}{h^2} - \frac{164891}{19488}\frac{\xi^3}{h^3} + \frac{47207}{19488}\frac{\xi^4}{h^4} - \frac{6229}{19488}\frac{\xi^5}{h^5} \right) y_{n+2}$$

$$
\begin{aligned}
&+\left(-\frac{9525}{812}\frac{\xi^2}{h^2}+\frac{\frac{3425}{609}\xi}{h}-\frac{4259}{1624}\frac{\xi^4}{h^4}+\frac{40819}{4872}\frac{\xi^3}{h^3}\right.\\
&\quad\left.-\frac{31}{1624}\frac{\xi^6}{h^6}+\frac{1801}{4872}\frac{\xi^5}{h^5}\right)y_{n+3}+\left(\frac{37563}{6496}\frac{\xi^2}{h^2}\right.\\
&\quad-\frac{170309}{38976}\frac{\xi^3}{h^3}+\frac{-\frac{4335}{1624}\xi}{h}-\frac{8539}{38976}\frac{\xi^5}{h^5}+\frac{57049}{38976}\frac{\xi^4}{h^4}\\
&\quad\left.+\frac{461}{38976}\frac{\xi^6}{h^6}\right)y_{n+4}+\left(-\frac{69}{56}\frac{\xi^2}{h^2}+\frac{\frac{39}{70}\xi}{h}+\frac{323}{336}\frac{\xi^3}{h^3}\right.\\
&\quad\left.-\frac{113}{336}\frac{\xi^4}{h^4}-\frac{1}{336}\frac{\xi^6}{h^6}+\frac{89}{1680}\frac{\xi^5}{h^5}\right)y_{n+5}+\left(\left(-\frac{15}{406}\xi\right)h\right.\\
&\quad+\frac{85}{3248}\frac{\xi^4}{h^2}-\frac{15}{3248}\frac{\xi^5}{h^3}+\frac{1}{3248}\frac{\xi^6}{h^4}-\frac{225}{3248}\frac{\xi^3}{h}\\
&\quad\left.+\frac{137}{1624}\xi^2\right)f_{n+6}
\end{aligned}
$$

(18)

Evaluating (18) at x_{n+j}, $j=6$ and its second derivative evaluated at $j=2,\ldots,5$, while its 1st derivative is evaluated at $j=0$ yields the following set of discrete equations, see table 3 for coefficients of the method.:

Table 1: Coefficients of the FOSBM which we called equations $13a-13d$

k	p	c_{p+2}	α_0	α_1	α_2	α_3	α_4	μ_K	β_0	β_1	β_2	β_3	β_4
4	3	$-\frac{2}{7}$	$\frac{11}{35}$	$-\frac{8}{5}$	$\frac{114}{35}$	$-\frac{104}{35}$	1	$\frac{1}{35}$	-	-	-	-	12
		$\frac{145}{444}$	$-\frac{13}{37}$	$\frac{63}{37}$	$-\frac{87}{37}$	1	-	$\frac{1}{37}$	-	-	-	35	-11
		$\frac{5}{468}$	$-\frac{1}{39}$	$\frac{7}{13}$	-1	$-\frac{19}{39}$	-	$\frac{1}{78}$	-	35	-	1	
		$-\frac{19}{252}$	$-\frac{421}{756}$	1	$-\frac{51}{84}$	$\frac{31}{189}$	-	$\frac{1}{42}$	-	-	-	1	

Table 2: Coefficients of the FISBM which we called equations $16a - 16e$

k	p	c_{p+2}	α_0	α_1	α_2	α_3	α_4	α_5	μ_K	β_0	β_1	β_2	β_3	β_4	β_5
5	4	$-\frac{137}{675}$	$-\frac{2}{9}$	$\frac{61}{45}$	$-\frac{52}{15}$	$\frac{214}{45}$	$-\frac{154}{45}$	1	$\frac{1}{15}$	-	-	-	-	-	4
		$\frac{391}{2595}$	$\frac{29}{173}$	$-\frac{176}{173}$	$\frac{438}{173}$	$-\frac{464}{173}$	1	-	$\frac{1}{173}$	-	-	-	-	108	-24
		$\frac{227}{17040}$	$-\frac{135}{15336}$	$\frac{1}{71}$	$\frac{141}{284}$	-1	$\frac{283}{568}$	-	$\frac{1}{284}$	-	-	-	135	-	3
		$\frac{1}{225}$	$-\frac{1}{30}$	$\frac{8}{15}$	-1	$\frac{8}{15}$	$-\frac{1}{30}$	-	$\frac{1}{5}$	-	2	-	-	-	
		$\frac{1399}{31920}$	$\frac{2015}{4256}$	-1	$\frac{969}{1064}$	$-\frac{67}{133}$	$\frac{509}{4256}$	-	$\frac{1}{266}$	-	-	-	-	-	3

Table 3: Coefficients of the SISBM which we called equations $19a - 19f$

k	p	c_{p+2}	α_0	α_1	α_2	α_3	α_4	α_5	α_6	μ_K	β_0	β_1	β_2	β_3	β_4	β_5	β_6
6	5	$-\frac{9}{58}$	$\frac{137}{812}$	$-\frac{243}{203}$	$\frac{1485}{406}$	$-\frac{1207}{203}$	$\frac{5265}{812}$	$-\frac{27}{7}$	1	$\frac{1}{203}$	-	-	-	-	-	-	45
		$\frac{13097}{154860}$	$-\frac{1955}{20648}$	$\frac{1739}{2581}$	$-\frac{21277}{10324}$	$\frac{8980}{2581}$	$-\frac{61891}{20648}$	1	-	$\frac{1}{5162}$	-	-	-	-	-	2436	-411
		$\frac{3262}{272595}$	$\frac{227}{18173}$	$-\frac{1492}{18173}$	$\frac{3386}{18173}$	$\frac{6424}{18173}$	-1	$\frac{9628}{18173}$	-	$\frac{1}{18173}$	-	-	-	9744	-	156	
		$-\frac{7}{10770}$	$-\frac{5}{1436}$	$\frac{37}{718}$	$-\frac{395}{718}$	1	$-\frac{2319}{4308}$	$\frac{29}{718}$	-	$\frac{1}{1077}$	-	-	-411	-	-	1	
		$-\frac{469}{86745}$	$\frac{361}{11566}$	[redacted]	1		$-\frac{55}{11566}$	$\frac{58}{5783}$	-	$\frac{1}{5783}$	-	-	-2436	-	-	-	6
		$-\frac{137}{4710}$	$-\frac{58997}{141300}$	1	$-\frac{3357}{2826}$	$\frac{1370}{1413}$	$-\frac{2601}{5652}$	$\frac{377}{3925}$	-	$\frac{1}{157}$	-	-	-	-	-	1	

IV. ANALYSIS OF THE BLOCK METHODS

Order and error constant

Following Fatunla [4, 5] and Lambert [7, 8 and 9], we define the local truncation error associated with the conventional form of (2) to be the linear difference operator:

$$L[y(x);h] = \sum_{j=o}^{k} \propto_j y(x+jh) - h^2 \beta_j y''(x+jh)$$

(20)

Where the constant coefficients $C_q, q = 0, 1 \ldots$ are given as follows:

$$C_q = \sum_{j=o}^{k} \propto_j$$

$$C_1 = \sum_{j=o}^{k} j \propto_j$$

.

.

.

$$C_q = \frac{1}{q!} \quad \sum_{j=o}^{k} j^q \propto_j - q(q-1)\sum_{j=o}^{k} j^{q-2}\beta_j$$

(21)

The new block methods 13, 16 and 19 are of uniform orders P = 4, 5 and 6 respectively (see tables1, 2 and 3). According to Henrici (1962), the block methods are consistent.

V. CONVERGENCE

Consider SISBM, the block methods shown in (19) can be represented by a matrix finite difference equation in the form:

$$IY_{w+1} = AY_{w-1} + h^2[\beta_1 F_{w+1} + \beta_0 F_{w-1}]$$

(22)

where

$$Y_{w+1} = (y_{n+1}, \ldots, y_{n+6})^T, \quad Y_{w-1} = (y_{n-5}, \ldots, y_n)^T,$$

$$F_{w+1} = (F_{n+1}, \ldots, F_{n+6})^T, \quad F_{w-1} = (F_{n-4}, \ldots, F_n)^T,$$

And w = 0, 1, 2, … and n is the grid index

And

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$B_1 = \begin{pmatrix} 0 & \frac{2641}{480} & -\frac{4991}{360} & \frac{3649}{240} & -\frac{959}{120} & \frac{475}{288} \\ 0 & \frac{33}{2} & -\frac{1772}{45} & \frac{637}{15} & -\frac{332}{15} & \frac{409}{90} \\ 0 & \frac{4599}{160} & -\frac{525}{8} & \frac{5643}{80} & -\frac{1467}{40} & \frac{1203}{160} \\ 0 & \frac{616}{15} & -\frac{4096}{45} & \frac{296}{3} & -\frac{256}{5} & \frac{472}{45} \\ 0 & \frac{5125}{96} & -\frac{8375}{72} & \frac{6125}{48} & -\frac{525}{8} & \frac{3875}{288} \\ 0 & \frac{657}{10} & -\frac{708}{5} & \frac{783}{5} & -\frac{396}{5} & \frac{33}{2} \end{pmatrix}$$

and $B_o = 0$

It is worth noting that zero-stability is concerned with the stability of the difference system in the limit as h tends to zero. Thus, as $h \to 0$, the method (13) tends to the difference system.

$$IY_{w+1} - AY_{w-1} = 0$$

Whose first characteristic polynomial $\rho(Q)$ is given by

$$\rho(Q) = \det(QI - A)$$

$$= Q^5(Q-1)$$
(23)

Following Fatunla [4], the block method (19) is zero-stable, since from (23), $\rho(Q) = 0$ satisfy $|Q_j| \leq 1,\ j = 1, \ldots, k$ and for those roots with $|Q_j| = 1$, the multiplicity does not exceed 2. The block method (19) is consistent as it has order P > 1. Accordingly following Henrici [6], we assert the convergence of the block method (19).

VI. STABILITY REGION OF THE BLOCK METHOD

To compute and plot absolute stability region of the block methods (19), the block method is reformulated as General Linear Methods expressed as:

$$\begin{pmatrix} Y \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} A & U \\ B & V \end{pmatrix} \begin{pmatrix} hf(y) \\ y_{i-1} \end{pmatrix}$$

where

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{900}{141300} \\ 0 & 0 & -\frac{4872}{11566} & 0 & 0 & 0 & \frac{12}{11566} \\ 0 & 0 & 0 & -\frac{1624}{4308} & 0 & 0 & \frac{4}{4308} \\ 0 & 0 & 0 & 0 & -\frac{9744}{18173} & 0 & -\frac{156}{18173} \\ 0 & 0 & 0 & 0 & 0 & \frac{9744}{20648} & -\frac{1644}{20648} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{45}{203} \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{45}{203} \\ 0 & 0 & 0 & 0 & 0 & \frac{9744}{20648} & -\frac{1644}{20648} \\ 0 & 0 & 0 & 0 & -\frac{9744}{18173} & 0 & -\frac{156}{18173} \\ 0 & 0 & 0 & -\frac{1624}{4308} & 0 & 0 & \frac{4}{4308} \\ 0 & 0 & -\frac{4872}{11566} & 0 & 0 & 0 & \frac{12}{11566} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{900}{141300} \end{pmatrix}$$

$$U = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ -\frac{13572}{141300} & \frac{65025}{141300} & -\frac{137000}{141300} & \frac{167850}{141300} & 0 & \frac{58997}{141300} \\ -\frac{116}{11566} & \frac{55}{11566} & \frac{5752}{11566} & 0 & \frac{6236}{11566} & -\frac{361}{11566} \\ \frac{174}{4308} & \frac{2319}{4308} & 0 & \frac{2370}{4308} & -\frac{222}{4308} & \frac{15}{4308} \\ \frac{9628}{18173} & 0 & \frac{6424}{18173} & \frac{3386}{18173} & -\frac{1492}{18173} & \frac{227}{18173} \\ 0 & \frac{61891}{20648} & -\frac{71840}{20648} & \frac{42554}{20648} & -\frac{13912}{20648} & \frac{1955}{20648} \\ \frac{27}{7} & -\frac{5265}{812} & \frac{1270}{203} & -\frac{1485}{406} & \frac{243}{203} & -\frac{137}{812} \end{pmatrix}$$

$$V = \begin{pmatrix} \frac{27}{7} & \frac{-5265}{812} & \frac{1270}{203} & \frac{-1485}{406} & \frac{243}{203} & \frac{-137}{812} \\ 0 & \frac{61891}{20648} & \frac{-71840}{20648} & \frac{42554}{20648} & \frac{-13912}{20648} & \frac{1955}{20648} \\ \frac{9628}{18173} & 0 & \frac{6424}{18173} & \frac{3386}{18173} & \frac{-1492}{18173} & \frac{227}{18173} \\ \frac{-174}{4308} & \frac{2319}{4308} & 0 & \frac{2370}{4308} & \frac{-222}{4308} & \frac{15}{4308} \\ \frac{-116}{11566} & \frac{55}{11566} & \frac{5752}{11566} & 0 & \frac{6236}{11566} & \frac{-361}{11566} \\ \frac{-13572}{141300} & \frac{65025}{141300} & \frac{-137000}{141300} & \frac{167850}{141300} & 0 & \frac{58997}{141300} \end{pmatrix}$$

Substituting the values of A, B, U, V into stability matrix and stability function,then using maple package yield the stability polynomial of the block method.Using a matlab program, we plot the absolute stability region of our proposed block method(see Fig. 2).

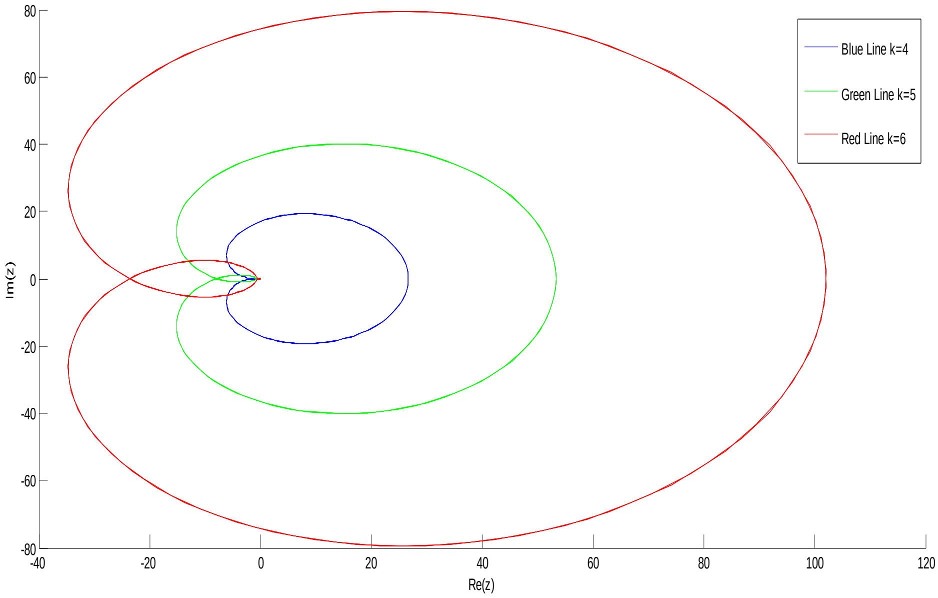

Fig. 1: *Absolute Stability Regions of the Discrete Methods*

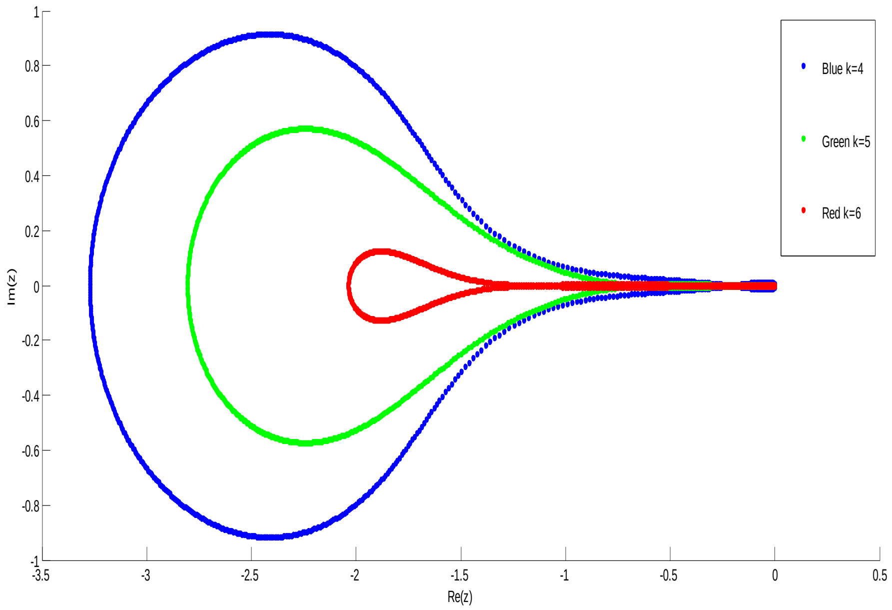

Fig. 2: *Absolute Stability Regions of the Block Methods*

In this paper, we use FOSBM, FISBM and SISBM to mean the Four, Five and Six Step Block Methods respectively.

VII. IMPLEMENTATION STRATEGIES

In this section, we have tested the performance of our four, five and six-step block method on two (2) numerical problems by considering two IVPs (Initial Value Problems). For each example, we obtained the absolute errors of the approximate solution.

Problem 1.1:
Consider the IVP for the step-size $h = 0.01$
$y'' - 100y = 0, y(0) = 1, y'(0) = -10$
Theoretical Solution given by: $y(x) = e^{-10x}$
Problem 1.2:
We consider the IVP for the step-size $h = 0.1$
$y'' + y = 0, y(0) = 1, y'(0) = 1$
Theoretical Solution given by: $y(x) = Cosx + Sinx\ x$

Table 4: Absolute errors for problem 1.1 using the FOSBM, FISBM and SISBM

x	*Absolute Errors* (FOSBM)	*Absolute Errors* (FISBM)	*Absolute Errors* (SISBM)
0	0	0	0.000e+0
0.01	1.1067e-5	1.2413e-6	1.353e-7
0.02	3.1403e-5	3.4226e-6	3.658e-7
0.03	5.2700e-5	5.7008e-6	6.051e-7
0.04	7.4521e-5	8.0308e-6	8.502e-7
0.05	8.2312e-5	1.0439e-5	1.104e-6
0.06	9.7067e-5	1.1244e-5	1.369e-6
0.07	1.1323e-4	1.2725e-5	1.450e-6
0.08	1.3052e-4	1.4369e-5	1.597e-6
0.09	1.3614e-4	1.6156e-5	1.763e-6
0.10	1.4725e-4	1.8102e-5	1.946e-6
0.11	1.6012e-4	1.8649e-5	2.099e-6
0.12	1.7459e-4	1.9725e-5	2.374e-6

where $Absolute\ Error = |y(x) - y|$

Table 5: Absolute errors for problem 1.2 using the FOSBM, FISBM and SISBM

x	$Absolute\ Errors$ (FOSBM)	$Absolute\ Errors$ (FISBM)	$Absolute\ Errors$ (SISBM)
0	0	0	0.000e-0
0.1	1.0368e-5	2.0448e-6	1.157e-7
0.2	2.9141e-5	5.6206e-6	3.099e-7
0.3	4.8219e-5	9.2394e-6	5.055e-7
0.4	6.6810e-5	1.2761e-5	6.957e-7
0.5	6.9493e-5	1.6149e-5	8.789e-7
0.6	7.3819e-5	1.8399e-5	1.054e-6
0.7	7.7560e-5	2.2224e-5	1.008e-6
0.8	8.0520e-5	2.5939e-5	9.226e-7
0.9	7.5308e-5	2.9389e-5	8.261e-7
1.0	6.5139e-5	3.2540e-5	7.216e-7
1.1	5.4005e-5	3.4422e-5	6.099e-7
1.2	4.2326e-5	3.7498e-5	4.919e-7

VIII. CONCLUSIONS

We have proposed a family of four, five and six-step block methods (FOSBM, FISBM, SISBM) with continuous coefficients from which multiple finite difference methods were obtained and applied as simultaneous numerical integrators ,without first adapting the ODE to an equivalent first order system. The methods were derived through interpolation and collocation procedures by the matrix inverse approach. We conclude that the new block methods are of uniform orders 3, 4 and 5 and were suitable for direct solution of general second order differential equations. All the block methods were self- starting and all the discrete equations used were obtained from the single continuous formulation including their derivatives which were evaluated at some interior points to form part of the block. The application of our block methods on two real life numerical problems (Problem 1.1 and Problem 1.2) give results which tend to converge to their respective theoretical solutions. Approximate solutions $y_1, y_2, \dots, y_k$ were also obtained in block at once thereby eliminating the use of any Predictors, this tend to speeds up the computational process. The absolute errors obtained from the application of our block methods to the problems stated (Table 1.1 and Table 1.2) shows the level of convergence and accuracy of our methods.

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