Advances in Agriculture, Sciences and Engineering Research
http://www.ejournal.sedinst.com
© Science Education Development Institute, 2013 Volume 3 (2) Feb.: 681 - 684, 2013
Printed in Nigeria **ISSN: 2276 - 6723**

ASSOCIATED PRIMES AND THEIR STRUCTURE AS SUBMODULES.

Ademola, Lois Adewoye.
Department of Mathematics, University of Jos, Jos, Nigeria.
Corresponding author: loisola@yahoo.com

ABSTRACT
Let R, be a ring , B ⊂ A, A a submodule and ρ a prime ideal of R. This paper considers the behaviour and structure of an associated prime ρ under specific condition.

KEYWORDS: prime ideal, associated prime, annihilator.

INTRODUCTION

In this paper we characterize the ring $R = \frac{K[x,y]}{(x^2,xy)}$, and the ρ- primary ideal with $\rho = (p,q)$ with the goal of finding richer structures of its extensions. We shall consider in particular (q^n) with respect to the ρ- primary ideal. Associated prime ideals have been known in commutative algebra for its focal contribution in the theory that is today called primary decomposition. Several research work have been built base on this fact some ofthese works include those of [1], [2], [3] and [4].

The failure of the getting a unique form of decomposition has made it halt the idea of a further research. Non the less the results obtained have turned out to have very intriguing contributions and this birthed the goal and purpose of this paper.

Basic Definitions and Preliminary Results

Definition 1:
An ideal ρ in a ring R is *prime* if $x, y \in \rho$ implies that one or the other (or both) of x, y is in ρ [5].

Definition 2:
Let R be a commutative ring with unit. An ideal ρ in R is *maximal* if there is no ideal strictly largerthan ρ (containing ρ) except R itself[5].

Theorem 1:
Let R be a commutative ring, and ρ an ideal. Then ρ is a prime ideal if and only if R/ρ is anintegral domain[5].

Definition 3:
If A, B are submodules of M, we define (A, B) to be the set of all $\rho \in R$ such that $\rho B \subseteq A$; it is an ideal of R. In particular, $(0, B)$ is the set of all $\rho \in R$ such that $\rho R = 0$; this ideal is called the *annihilator* of R and is also denoted by $Ann(R)$ [5].

Definition 4:
An ideal ρ in a ring R is primary if $\rho \neq R$ and if $xy \in \rho$ then either $x \in \rho$ or $y^n \in \rho$ for some $n > 0$ [5].

Proposition 1:
Let ρ be a primary ideal in a ring R. Then $r(\rho)$ is the smallest prime ideal containing ρ. If $\rho = r(\rho)$, then ρ is said to be ρ -primary [5].

Definition 5:
Let B be a right R -module. An ideal ρ of R is called an *associated prime* of B if there exists a prime submodule $B \subset A$ such that $\rho = ann(B)$. The set of associated primes of A is denoted by $ass(A)$[6].

3.Proof of Main Result
Claim 1:
$(q^n) \neq 0$ in R, $q^n \in R$.

Proof of Claim 1:
The image of y in $R = \frac{K[x,y]}{x^2,xy}$ is obviously q. We shall prove this claim by contradiction as follows:
Suppose $q^n = 0$ then this is true if and only if $y^n \in (x^2, xy)$,$q^n \in R$.

But then this mean that $y^n = x^2 f + xyg$, for some $f, g \in K[x, y]$. So then, $y^n \in (x \quad)$, which is a contradiction since $y^n \notin (x)$.

We will next show that q^n is truely ρ- primary. To do this, we would have to show that the Associated prime ideal; $Ass_R\left({}^{R}/_{(q^n)}\right) = \{\rho\}$. Thus we have to show in particular that:

i. $\rho \in Ass_R\left({}^{R}/_{(q^n)}\right)$, where $\rho = (p, q)$.
ii. If $\varrho \in Ass_R\left({}^{R}/_{(q^n)}\right)$, then $\varrho = \rho$ for all $\varrho \in Ass_R\left({}^{R}/_{(q^n)}\right)$.

Proof of Claim 1.i:
The image of x in $R = \frac{K[x,y]}{(x^2,xy)}$ is obviously $p, p \in R$. In other to continue this proof we make a claim:

Claim 2:
$p \neq 0$ in R.

Proof of Claim 2:
Proving by contradiction, and taking the similar steps as in the prove of Claim 1, we have:
Suppose $p = 0$, then this will be true if and only if $x \in (x^2, xy)$,in k[x, y]. This consequently means that $x = x^2 f + xyg$, for some $f, g \in K[x, y]$. So then, $1 = xf + yg$, thus $1 \in (x, y)$, which is a contradiction since $1 \notin (x, y)$.

Thus we have that $p \neq 0$ in R.
We will next show that $\rho = Ann_R(\bar{p})$, where $\bar{p} \in {}^{R}/_{(q^n)}$.
Now by definition, $Ann_R(\bar{p}) = \{r \in R | r\bar{p} = 0\}$, so to show $\rho \subseteq Ann_R(\bar{p})$, we would need to sow that $p, q \in Ann_R(\bar{p})$.

Claim 3:
$p, q \in Ann_R(\bar{p})$.

Proof of Claim 3:
Choose $p \in R$, and $\bar{p} \in {}^{R}/_{(q^n)}$, we would have that:

$$p\bar{p} = \overline{pp} = \overline{p^2} = 0$$

Since $p^2 = 0$ in $R = \frac{K[x,y]}{(x^2,xy)}$.

Similarly,we choose $q \in R$, and $\bar{p} \in {}^{R}/_{(q^n)}$, we would have that:

$$q\bar{p} = \overline{qp} = \overline{pq} = 0$$

Since $pq = 0$ in $R = \frac{K[x,y]}{(x^2,xy)}$.
Thus $p, q \in Ann_R(\bar{p})$.
Hence $\rho \subseteq Ann_R(\bar{p})$.
In fact $\rho = Ann_R(\bar{p})$, since:

Claim 4:

i. ρ is maximal,

ii. $Ann_R(\bar{p}) \neq R$.

Proof of Claim 4.i:
To show maximality, we have that :

$$R/\rho = \frac{k[x,y]/(x^2,xy)}{k[x,y]/(x,y)} = k[x,y]/(x,y) = k.$$

So R/ρ is a field, meaning ρ is maximal.

Proof of Claim 4.ii:
$Ann_R(\bar{p}) \neq R$, since $1 \notin Ann_R(\bar{p})$. That is :
$1\bar{p} = \overline{1p} = \bar{p}$.

Claim 5:
$\bar{p} \neq 0$in $R/(q^n)$.

Proof of Claim 5:
Proving by contradiction, and taking the similar steps as in the prove of Claim 1, we have:
Suppose $\bar{p} = 0$, then this will be true if and only if $p \in (q^n)$,in k[x, y]. Thus $p + q^n h = 0$, for $h \in K[x, y]$. So then, $x + y^n h \in (x^2, xy)$,meaning $x|h$. So $h = xh_0$.
Thus $x + y^n h \in (x^2, xy)$, becomes $x + y^n xh_0 = x^2 f + xyg$ which implies that
$1 + y^n h_0 = xf + yg$. So that $1 \in (x, y)$, which is a contradiction so $q \neq 0$ in $R/(q^n)$.

The proves of Claims 2, 3, 4 and 5 proves Claim 1.i.

Next we shall prove Claim 1.ii.

Proof of Claim 1.ii:
Let $\varrho \in Ass_R\left(R/(q^n)\right)$, then $\varrho \supseteq (q^n)$ and since ϱ is prime then it follows that$\sqrt{\varrho} = \varrho \supseteq \sqrt{(q^n)}$.

Claim 6:
$(q^n) \supseteq \rho^{n+1}$.

Proof of Claim 6:
Recall that $\rho = (p, q)$, so $\rho^{n+1} = (p, q)^{n+1} = (p^{n+1}, p^n q, \dots, pq^n, q^{n+1})$ and since $p^2 = 0 = pq$, it follows that $\rho^{n+1} = (q)^{n+1}$. So $(q^n) \supseteq (q)^{n+1}$and $\supseteq \sqrt{(q^n)} \supseteq \sqrt{(q)^{n+1}} \supseteq \sqrt{\rho^{n+1}}$.
So $\varrho \supseteq \rho$.

Thus by the maximalityof ρ, $\varrho = \rho$.

Thus proving Claim1.ii.

CONCLUSION
With the prove of Claim 1 and the various extensions of the ring $R = \frac{K[x,y]}{(x^2,xy)}$ to various properties governing its structure, we have been able to further ascertain that the any given ring can be decomposed to show that it has richer structures and that associated prime ideals play a vital role in the structure of Rings.

REFERENCES

[1] S. Annin,(2002). Associated primes over skew polynomial rings.

[2]C. Faith, (2000). Associated primes in commutative polynomial rings.

[3] A. Facchini (1998). Endomorphism Ring and Direct Sum decompositions in some clases of modules. Progress in mathematics, Volume 167.

[5]M. F. ATIYAH and I. G. MACDONALD, (1969).Introduction to Commutative Algebra

[6]. T.Y. Lam(1998).Lectures on Modules and Rings