UNIVERSITY OF JOS

PROGRESS IN THE NUMERICAL TREATMENT OF STIFFNESS

INAUGURAL LECTURE

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By

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Inaugural lecture: Prof. P. Onumanyi

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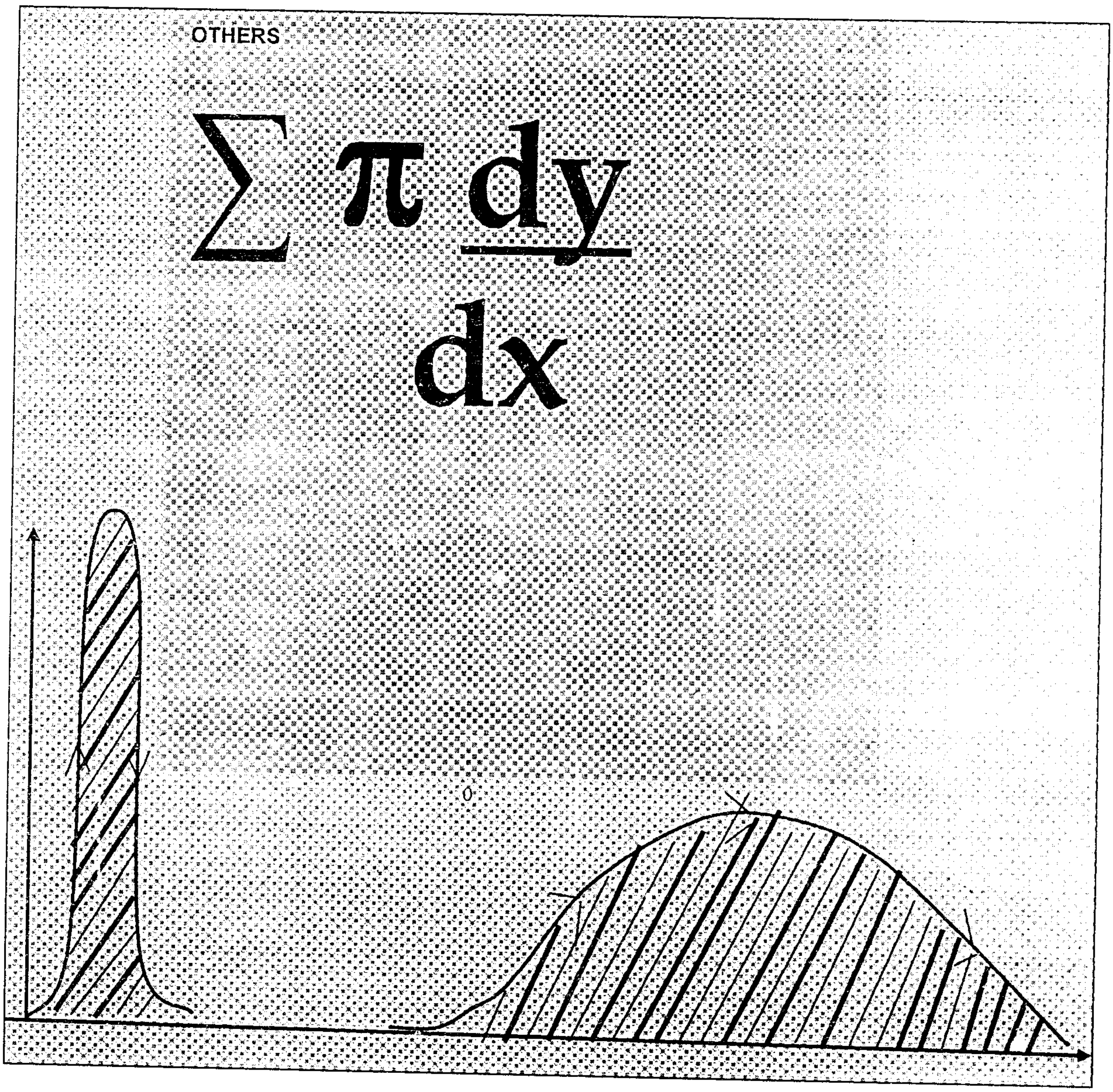
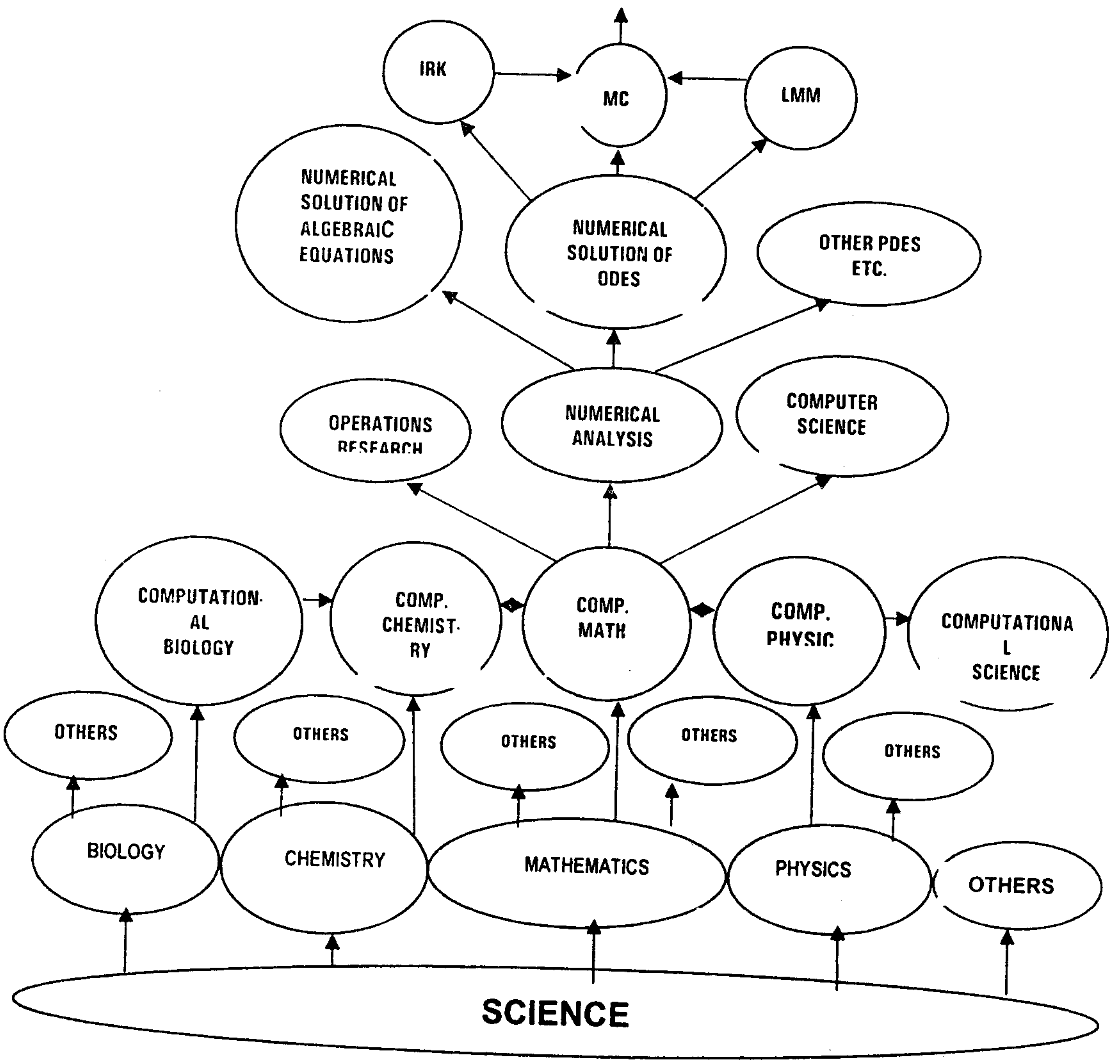


WAY FORWARD?


IRK
MC
LMM
NUMERICAL SOLUTION OF ALGEBRAIC EQUATIONS
NUMERICAL SOLUTION OF ODES
OTHER PDES ETC.
OPERATIONS RESEARCH
NUMERICAL ANALYSIS
COMPUTER SCIENCE
COMPUTATION-AL BIOLOGY
COMP. CHEMIST-RY
COMP. MATH
COMP. PHYSIC
COMPUTATIONA L SCIENCE
OTHERS
OTHERS
OTHERS
OTHERS
OTHERS
BIOLOGY
CHEMISTRY
MATHEMATICS
PHYSICS
OTHERS
SCIENCE

Points about fig. 1:

(a) The knowledge of science is growing and rapidly too. We illustrate this fact by the growth tree of science in fig. 1

(b) Computational Science has emerged as a new science to which I belong by training and research.

(c) The paper conjectures that the Multi-step Collocation (MC) is the way forward for the numerical solution of stiff ODES.

1 INTRODUCTION

1.0 TITLE OF LECTURE

Stiffness in common usage generally indicates a more difficult and severe situation than usual. In Numerical analysis it is an important mathematical problem with many applications to real life. We will give some in section 5 for illustration in dynamics. A basic difficulty that arises in the numerical treatment (solution) of a stiff problem is that of numerical stability.

The traditionally successful numerical methods such as Adams Moulton and the explicit Runge-Kutta methods suffer step size constraint imposed by stability when applied to stiff problems. Thus, to overcome this stability restriction on the stepsize, numerical methods that possess unbounded regions of absolute stability (A-stable methods) have been recommended for the solution of stiff problems.

The A- stability requirement on numerical methods as you will see in this lecture, is a severe demand on numerical methods to achieve. The historical appearance of stiffness in the Numerical Analysis literature was the paper by curtiss and Hirschfelder (1952). More than its importance then, the challenges it posed to numerical method led to its intensive study and research by the Numerical Scientists. "Around 1960, things became completely clear that the world was full of stiff problems." according to Dahlquist in Aiken (1985). In the same reference, Aiken said, "stiff problem is not a mere curosity..."

In the 1977/78 M.Sc. degree class at Imperial College London, I received ten (10) hours of lectures on stiff Initial value problem by Prof. J. Cash. You will find in my Ph.D thesis (1981) stiff and singularly. Perturbed problems. Among my scientific publications, and PhD thesis supervised by me you will find stiff problems. Mr. vice-chancellor sir, you can then appreciate my choice of title for today's lecture.

The main thesis of this lecture is that there exist continuos linear multi-step methods including the hybrid ones that are based on a general matrix inversion approach. From the constructed continuous interpolant, we obtain block methods and block hybrid methods, which are all special cases of the general linear methods with improved acceptable stability and high accuracy properties for practical applications to stiff problems.

However, before we proceed to stiffness let me share a few thoughts with you on growth of science (see fig 1).

1.1 IMPACT OF COMPUTER AND INFORMATION AGE ON SCIENCE DEVELOPMENT.

The rapid progress in information technology and the increasing power of computer systems and scientific computing offer mankind a potential and new perspective for solving complex problems. For this reason there remain few areas of human intellectual activity that have not been shaped significantly by the expanding influence of mathematics through the use of mathematical models and computer simulation which have become important tools in scientific investigations, among other applications. As an example, in 1998 one of the winners of the Nobel Prize in chemistry was a computational chemist. The prize was awarded for his work in computer modeling and simulation of chemicals and chemical processes.

Today, there are three major categories of Scientists within each science discipline: Experimental Scientists, Theoretical Scientists, and Computational Scientists. In Mathematics we now have Pure Mathematicians, Applied Mathematicians and Computational Mathematicians (see fig.1) above. Mathematics not only expresses relationships and laws of nature but compute real, relative numerical values that must have meaning and some useful interpretation. Today, multi-disciplinary approaches to problems have contributed to the emergence of modern computational Science.

A natural question that can arise is why computational mathematics? Closed form or generalized analytic solutions are desired for mathematical problems. Whenever such is not feasible, approximate methods are employed. This could lead to further analytic and numerical methods that fully describe the solution to the given problem. These descriptive and number crunching procedures suitable for computer assistance constitute computational methods. In year 2001, a new journal of the society for the foundations of computational mathematics was born and it includes papers from biology, chemistry, physics, psychology, linguistics, political science, computer science, cryptology, economics, engineering, visualization, and so on and so forth. New and recent titles in mathematics have appeared in the Springer series on computational science and engineering like: "Mathematics-key technology for the future"

(2003); "Mathematics unlimited 2001 and beyond" and Progress in industrial mathematics at European consortium for mathematics in Industry (ECMI) 2000".

In all these, efficient transfer between science and society is being shown to be crucial for their future development and researches, and range from automotive industry to computer technology and medical visualization.

In general, the projects of the new mathematical ideas and methods being proposed in these titles are decisive for the solution of industrial and economic problems. Mathematics, imbedded in an interdisciplinary concept, has become a key technology. We will illustrate this point later in this lecture under applications.

It is in this spirit of moving mathematics into partnership with the society in Nigeria that the present administration of the National Mathematical Centre (N.M.C), Abuja is currently pursuing collaboration with some selected Federal Universities in a Joint Higher Degree Programme (JHDP) as a form of linkage in areas like mathematics in Biomedicine, mathematical modeling and simulation, mathematics of finance, communications and Information systems among others.

So I am saying in this preamble that Nigerian Scientists should go into collaborative work more than ever before, making the best use of the computer and mathematical modeling and simulation techniques, which is the trend world over.

1.2 APPRECIATION

My appreciation goes to my family, my better half Mrs. Bosede Onumanyi who has been a close associate to me and true companion and friend, my children Jumoke the beloved, who after sleeping on my chest each night for the first six months of her life refused to sleep thereafter on the bed, Adeiza the baba, Ohunene my heart, Albert the great, Seyi Inya Chundun and Mary-Blessing, Inya Owewu. I love you all and I thank you for coping with my books all over the house.

I thank Professor Len Liverpool (Professor of complex analysis) who together with the cooperation of my long standing friend and brother, Professor M.S. Audu (Professor of abstract algebra) who made my movement to the University of Jos possible in 1992.By our efforts, and the grace of the Almighty God, we all can see clearly the development of mathematics department and we give God the glory. I thank Professor C.O.E. Onwuluri for being a friend from the NYSC days in Abeokuta in 1975/76 till today.

The present acting Head, Dr. S.U. Momoh has been maintaining the continuity of the good things of the department. He has been a true junior brother to me by his overwhelming love and respect. You will grow to be respected too. Amen.

Now I will give a few definitions of some terminology, which we will use in the presentation.

1.3 TERMINOLOGY

A **MATHEMATICAL PROBLEM** consists of a set of equations in which you will find some known variables and some unknown variables.

Solving the mathematical problem on the computer for the unknown variables is **SCIENTIFIC COMPUTING**.

NUMERICAL ANALYSIS is the branch of mathematics involved with scientific computing for obtaining numerical values (solutions) to various mathematical problems for which there are always physical (real-life) applications. It involves the study, development and analysis of algorithms.

A basic course in numerical analysis contains such mathematical problems as:

Errors and their propagation
Finite Differences (FD) and function approximations
Numerical differentiation and integration
Numerical solutions of ordinary differential equations (ODE's)
Numerical solutions of partial differential equations (PDE's)

NUMERICAL TREATMENT of a mathematical problem is the numerical solution (remedy) to a mathematical problem.

The accuracy of the numerical method is indicated by the order. The higher the order the better the accuracy.

A **COMPUTER PROGRAM** (**routine** or **code**) is a set of instructions written in a language to communicate the algorithms developed to the computer. Afterall the algorithms that we study or develop invariably are designed for use on high-speed computers and therefore before the solution to a mathematical problem can be obtained a computer code is a crucial intervention step, to provide discrete or continuous output of numerical results for the numerical scientist. For instance, Fortran (Formula Translation) is the oldest programming

Language. A collection of such programs is software, which is a topic now best left out of the science of Numerical analysis per se. Mathematical software, like Numerical analysis, is now a branch of its own in computational science and engineering.

We have in our mathematics-computing Laboratory several such softwares, **MAT LAB, MAPLE** and **SCIENTIFIC WORKSPACE,** among others.

Nowadays there are many computers and computer languages for various types of jobs and it is always advisable to consult an expert for the most appropriate language to choose for the solution of your problem. Come to us for the solution of your mathematical problems at least for advice.

Next we consider the mathematical problem to which my research focus has been, in the past twenty-three (23) years when I obtained my PhD and for the past twelve (12) years since I have been professing.

We are now going into the main menu. Please tighten your sit belts for the flight is about to take-off. There will be jargons of our trade that we cannot avoid in the flight.

2. ODE PROBLEMS AND THEIR SOLUTIONS BY ONUMANYI'S MULTI-STEP COLLOCATION FORMULA

2.1 ODE PROBLEMS

A mathematical problem involving one or more derivatives of an unknown variable (dependent variable) x in the equation is called a differential equation (DE). It can be linear or non linear. It is linear if the dependent variable x has a linear combination of the independent variable t, otherwise it is called non linear.

If $x = x(t)$, which involves a single independent variable t, then the equation is called ordinary DE (ODE), otherwise it is called partial DE (PDE).

One of the most frequently occurring mathematical problems in Numerical analysis is the solution to the initial value problem (IVP) or boundary value problem (BVP) in the n first order ordinary differential equations

IVP: $x' = f(t,x)$, $x(t_0) = A$ given (1)

BVP: $x' = f(t,x)$, $x(t_0) = A$, $x(t_{end}) = B$ given,

Where, $f(t,x)$ is a given function and $t_0 \leq t \leq t_{end}$ is the interval or range for which the solution $x(t)$ to the system (1) is required by a numerical integration procedure.

For systems of equations, x and f are vectors.

The first order ODE (1) covers all types of ODE's but sometimes we take advantage of certain higher order (n >1) ODE's and solve them directly without employing their reduction to (1).

A numerical scientist is using computer routines from a mathematical software library to obtain a sequence of numerical values $\{x_i\}$ to the solutions $\{x(t_i)\}$ at points $\{t_i\}$. He wants to know whether or not his numerical values $\{x_i\}$ are sufficiently accurate. The measure of the accuracy of his output values x_1, - - -, x_{end} is given as

$$E_i = x_i - x(t_i),\ t_0 \leq t_i \leq t_{end} \qquad (2)$$

Equation (2) is called global errors (errors), which most often are not known, and their estimates are all we settle for. This is in contrast to the local error e_i at the point $\{t_i\}$, (see fig.2).

It has been established that E_i in (2) is linked to e_i shown in fig.2. The local errors e_{i+k} are always within his reach at each step of the computations in a code and he tries to control it to meet all the challenges of the problem at hand. Most decisions by the automatic codes are based on how small or large e_{i+k} is relative to a tolerance (Tol) given by the user. It is then hoped or expected to give a good global error at each step t_{i+k}.

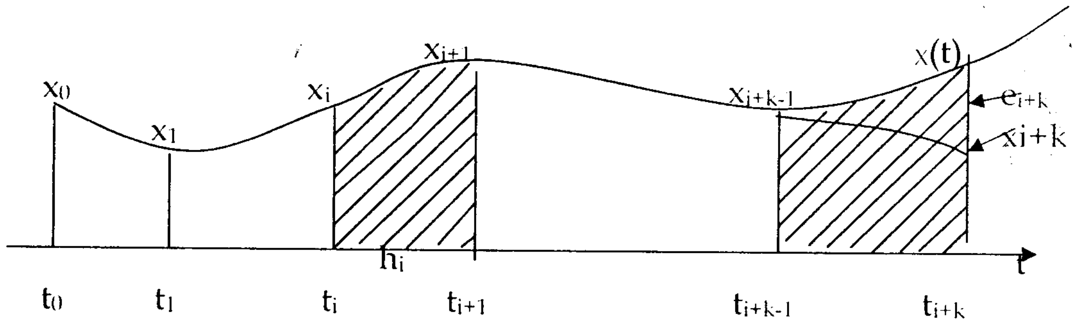


Fig.2 Illustration of the one step (k=1) and multistep methods (k-steps).

The single step $t_i \leq t \leq t_{i+1}$ and the k^{th} -step are shaded in fig.2.
The variable step size $h_i = t_{i+1} - t_i$ can be constant if $h = h_i$, for all values of i.

A linear multi-step method can be represented as a k-step, formula of the form

$$\sum_{j=0}^{k} \alpha_j x_{i+j} = h \sum_{j=0}^{k} \beta_j f_{i+j} \qquad (3)$$

with the constants α_j and β_j given. When $\beta_k = 0$, x_{n+k} is readily obtained and (3) is called explicit, otherwise it is implicit. Important specific classes in (3) are the Adams, BDF, and Simpson methods, with different ways of obtaining them.

2.2 ONUMANYI'S MULTI-STEP COLLOCATION FORMULA.

Let r denote the number of interpolation points and s denote the number of distinct collocation points to be used.

One formulation of Multi-Step Collocation (originally proposed by Lie and Norsett 1989) that leads directly to a matrix inversion is the following

$$X(t) = u^T C^T v(t), \quad t_i \le t \le t_{i+k} \text{ and } x_{i+k} \equiv x(t_{i+k}) \qquad (4a)$$

Where $v(t) = (v_1(t),---,v_{r+s}(t))^T$ is a column vector of r+s rows consisting of the given r+s basis functions of t. Throughout, except where it is considered as a special case, $v_i(t) = t^{i-1}$ is used in the rest of this presentation (power basis or Taylor polynomial basis) because polynomials are the bedrock of numerical methods and

$u = (x_{i+1}, ,x_{i+r}, f_{i+1},----, f_{i+s})$, with $C \equiv D^{-1}$ as the unique inverse matrix of D

$$D = \begin{pmatrix} v_1(t_i) & v_{r+s}(t_i) \\ v_1(t_{i+1}) & v_{r+s}(t_{i+1}) \\ * & * \\ * & * \\ v_1(t_{i+r-1}) & v_{r+s}(t_{i+r-1}) \\ v'_1(C_i) & v'_{r+s}(C_i) \\ v'_1(C_{i+1}) & v'_{r+s}(C_{i+1}) \\ * & * \\ * & * \\ v'_1(C_{i+s-1}) & v'_{r+s}(C_{i+s-1}) \end{pmatrix}^T \qquad (4b)$$

The matrix D is of the extended Vander monde type which by Onumanyi et.al. (2002) is shown to be invertible (non-singular).

The formula 4a and 4b is a theoretical result and it is not suitable as an algorithm for a direct application. This is because of the evaluation of C needed at every step of the integration process by the MAPLE, which is a non-trivial

thing. Instead C is evaluated once by the MAPLE software and the r.h.s of (4a) is fully expanded out and simplified with rearrangements of the terms, then the solution x(t) is readily written as

$$X(t) = \sum_{j=1}^{r} \Phi_j(t)\, X_{i+j-1} + h \sum_{j=1}^{s} \Psi_j(t)\, f_{i+j-1}, \qquad t_i \leq t \leq t_{i+k} \tag{5}$$

where the continuous coefficients $\Phi_j(t)$ and $\Psi_j(t)$ in (5) are explicitly determined in terms of the columns of the inverse matrix C and the basis vector function v(t) specified. The vector v(t) can be any of the polynomials, Lagrange, Taylor, Ortiz recursive formulation of the canonical polynomials of Lanczos, special functions like the exponential functions, Chebyshev and Legendre cubic splines and so on, as may be deemed appropriate to choose in a particular application. This is a unique flexibility in this formulation of the multi-step collocation.

Hence formula (5) is a computational algorithm which is better suited for step- by- step calculations than (4a) and (4b).

The formula (4a) with the inverse matrix C determined explicitly by a MAPLE code is Onumanyi 's formula.

In the theory and computation of solutions to system (1), the formula (4a) and (4b) is the first such general algorithm proposed. It has the ability to unify into itself many specific classes of important numerical methods as follows

i) One-step methods in discrete/continuous forms
Explicit Euler, implicit Euler, implicit trapezoidal, Gauss methods, Radau methods, Lobatto methods and the trapezoidal method using exponential basis (Fatokun 2002) for an order three L-stable method).

ii) Multi-step methods in Discrete / continuous forms
Simpson, explicit and implicit Adams, Backward Differentiation Formulae (BDF), Hybrid methods unlimited.

Because the formula (4a) and (4b) is naturally continuous it provides several benefits for the computation of the IVP namely, dense output, error estimation, and automatic control of step size in the Adams codes.
The extension of (5) to the IVP of the special second order ODE

$$x'' = f(t,x),\ x(t_0) = x_0,\ x'(t_0) = y_0 \qquad (6)$$

Gives a similar formula

$$x(t) = \sum_{j=1}^{r} \Phi_j(t)\, x_{i+j-1} + h^2 \sum_{j=1}^{s} \Psi_j(t)\, f_{i+j-1} \qquad (7)$$

In the formula (4a) and (4b) x' is merely replaced throughout by x'' including the entries of the matrix D of equation (4b). For the equation (6) and by the use of first derivative continuity condition (C' - approximation) it was possible to construct from (7) by evaluation, a Block Numerov Method (BNM) algorithm of order four which is self-starting and does not require Predictor - Corrector formulation. The new parallel algorithm has ability to solve a BVP directly. This was one result outside stiff problems that we thanked God for.

3. STIFFNESS, LINEAR STABILITY THEORY AND REGION OF ABSOLUTE STABILITY.

STIFFNESS

An illustration of a characteristic representation of stiffness is shown in figure 3 below

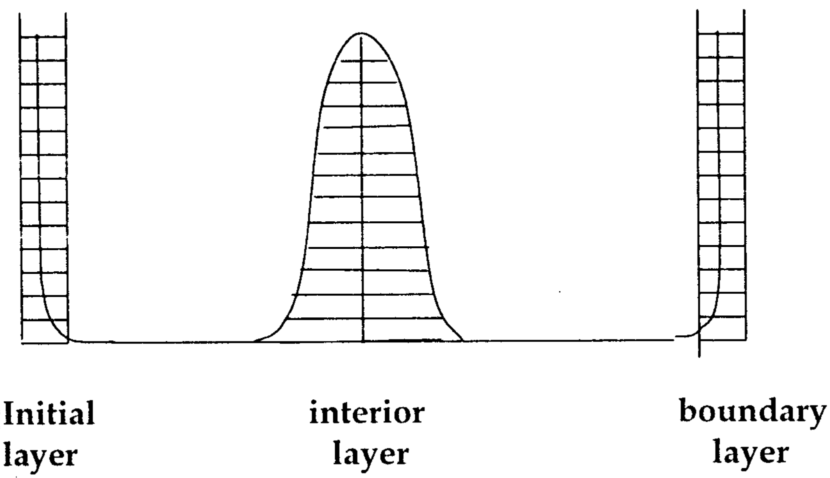


Fig.3

We shall adopt a verbal definition of stiffness, valid for both linear and non-linear problems.

DEFINITION

If a numerical method is forced to use, in a certain interval of integration, a step size, which is excessively small in relation to the smoothness of the exact solution in that interval, then the problem is said to be stiff in that interval. Unlike the traditional linear definition of stiffness, our definition allows a single equation, not just a system of equations, to be stiff. It also allows a problem to be stiff `in parts': a non-linear problem may start off non-stiff and become stiff, or vice versa. It may even have alternating stiff and non- stiff intervals.

TRADITIONAL LINEAR STABILITY THEORY.

Associated with a linear stiff system of size n of the form (1) are n complex eigenvalues with negative real parts of the Jacobian of the system, and the sizes of the smallest and the largest real parts are widely apart.

Now if you arrange the sizes (magnitude or modulus) of the real parts of the eigenvalues from the smallest size to the largest size then the ratio of the largest and the smallest is called the stiffness ratio which gives the measure of stiffness in the ODE problem (1).

This definition of stiffness is not valid for non-linear systems (1). It is based on the linear model and its eigenvalues.
As an example of a linear stiff problem, consider the equation.
$x'' - 1001x' + 1000x = 0$
We can write this as $x' = Ax$

$$A = \begin{pmatrix} 0 & 1 \\ -1000 & -1001 \end{pmatrix}$$

So the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = -1000$. The equation has solution

$x = Ae^{-t} + Be^{-1000t}$. The stiffness ratio in this example is 1000 showing stiffness with transient given by e^{-t} and e^{-1000t}, which is, slow and fast transient respectively.

From all I have said you can see that stiffness is a mathematical problem associated (inherent) with the ODE.

Stiffness is not associated with the numerical method (algorithm) used to obtain the solution of the ODE. But because of the scourging effect on most known and successful numerical methods before 1952, nowadays a numerical analyst will ask (including automatic codes) whether the ODE problem (1) to be solved (numerical treatment) is "stiff" or "non-stiff" before proceeding to choose the method to adopt.

It is just like a medical doctor. He will examine the patient first for the type of ill health before recommending an adequate medical treatment. Similarly, a lawyer too will investigate his client first before deciding on a course of legal procedure to follow.

Figure 4 shows a characteristic behaviour of stiffness in a natural physical situation.

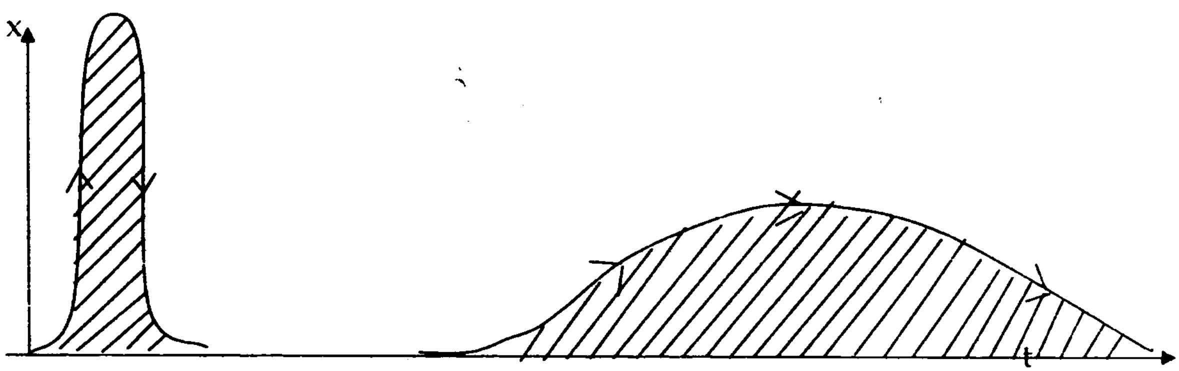


Fig 4. Illustration of stiff and non-stiff phenomenon by Adamawa hills in Nigeria.

REGION OF ABSOLUTE STABILITY AND ACCEPTABLE NUMERICAL METHODS FOR STIFF PROBLEMS.

Having identified that the ODE is stiff (that is, the stiffness ratio is large) one has to seek a method of solution to the stiff problem from the acceptable numerical (A-stable) methods.

Using the linear stability theory for multi-step methods leads to the famous important theorem of Dahlquist (1963) which says that A-stable multi-step methods cannot have high order greater than two which is the Trapezoidal method.

Attempts to circumvent this barrier proceeded mainly in two directions: either study methods with slightly weaker stability requirements that is, "nearly" A-stable multi-step methods, A(α)- stability and stiff stability or introduce new classes of methods.

The second option to strengthen the method in new ones will occupy our attention in the following sections to provide A-stable multi-step methods (block Hybrid) for stiff problem.

WHY A-STABLE METHODS?

This question arises because some convergent methods (which are zero-stable and consistent) fail to be useful in practice, particularly with stiff problems. For such methods (all explicit methods are inclusive), very small step sizes h, in (3) are necessary for them to solve the stiff problem. This is not practicable and hence unacceptable for good scientific computing. While the issue of convergence is for when the step size h=0 (zero-stability requirement), in practice, h is small but not equal to zero. This is where the difficulty is hinged leading to the concept of absolute stability.

The Absolute stability definition ensures that all eigenvalues in absolute sense have their real parts contained within a unit disc (circle). The graphical representation of this concept leads to absolute stability region, which we define as the set of points of the complex plane (x-t) for which the numerical method is absolute stable. These regions are symmetric about the horizontal axis (t) and they have implications on the acceptable choice of step size h in (3). This is to ensure the stability of the numerical solution away from the transient phase as the calculations progress forward.

A - STABILITY

A numerical method is said to be A-stable if its absolute stability region contains the whole of the left hand half-plane. That is, h>0 for all values of Re (hλ) is negative. This does not impose any restriction on the choice of h.

We give a few examples for illustration.

1. The explicit (Euler) method

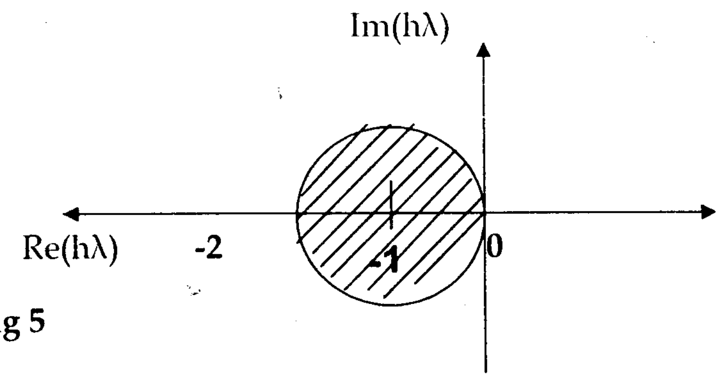


Fig 5

$0 < h < -2 / \text{Re}(\lambda)$

$0 < h < 2 / |\text{Re}(\lambda)|$

2. The implicit Euler (Backward Euler) method

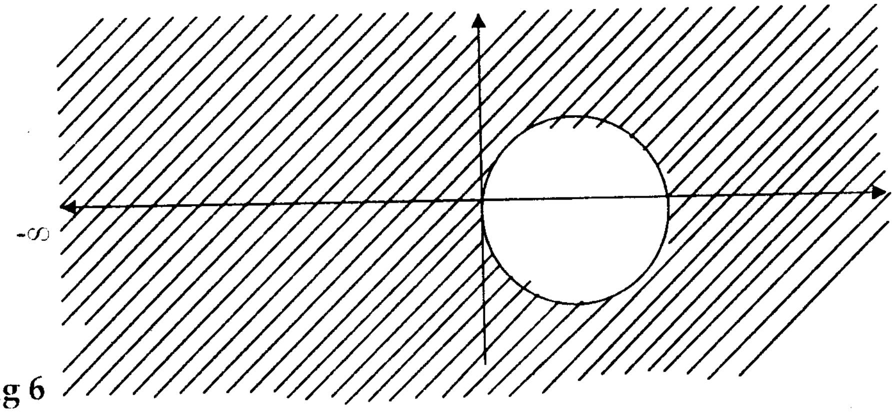


Fig 6

$0 < h < -\infty / \text{Re}(\lambda)$

$0 < h < \infty$

This is true of A–stable methods.

That is, there is no restriction on the choice of step size h.

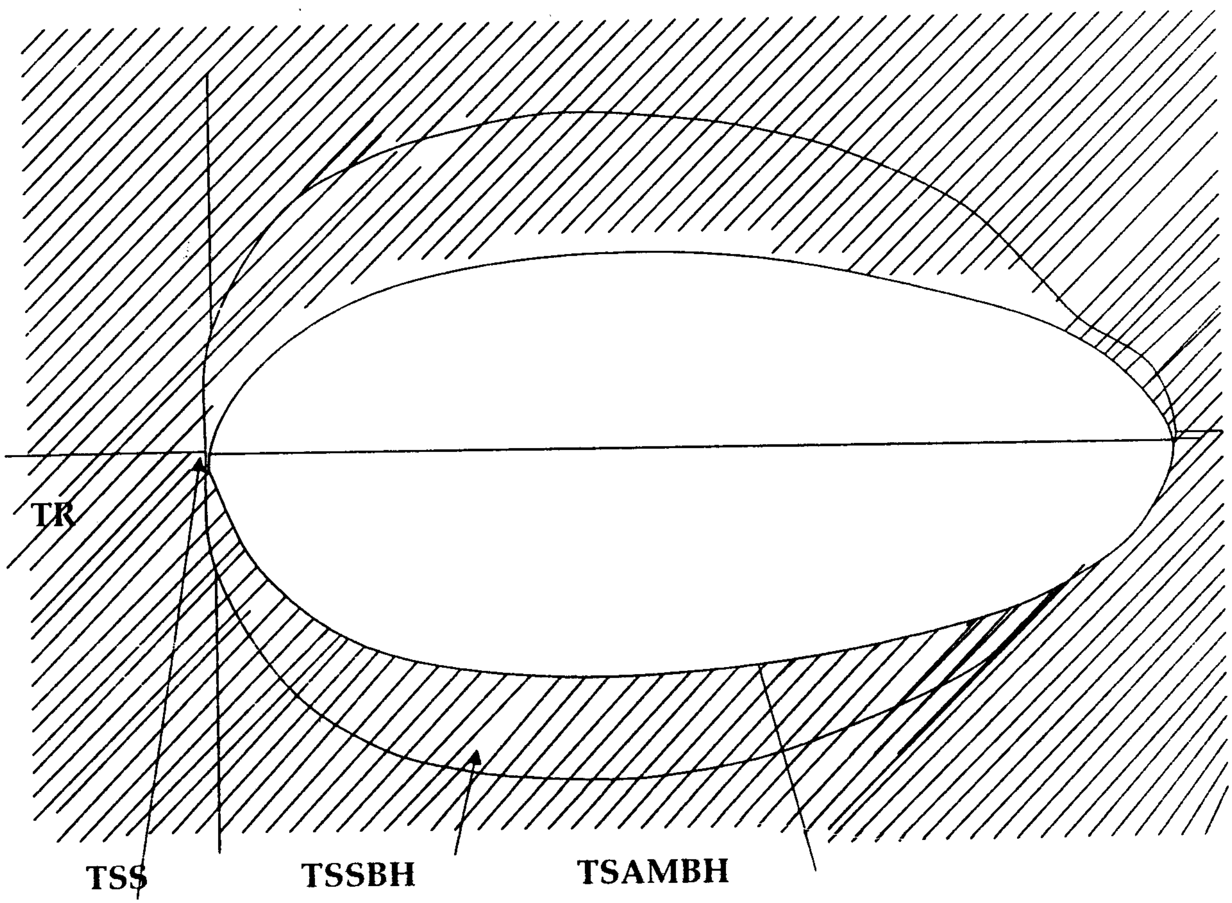


Fig 7

Table 1

Errors of methods for various stiffness ratios applied to the test equation $x' = \lambda x$.

$\lvert\lambda\rvert$	TSSM Non-stiff method order four	TR Precisely A-stable order two	TSAMBH A-stable order four	TSSBH A-stable order four
1	2.0 x 10-7	2.3 x 10-4	1.2 x 10-5	1.5 x 10-7
10	5.5 x 10-1	2.0 x 10-9	1.1 x 10-9	1.1 x 10-10
100	Diverged	3.0 x 10-5	8.0 x 10-9	1.1 x 10-9
1000	Diverged	4.5 x 10-1	4.6 x 10-7	2.1 x 10-8
1000000	Diverged	1.0	2.0 x 10-1	1.2 x 10-4

TSSM – Two step Simpson method

TR – Trapezoidal rule method

TSAMBH – Two-step Adams Moulton Block Hybrid

TSSBH – Two-step Simpson Block Hybrid

In the table 1 shown, the errors committed by various numerical methods applied to the test equation for stiffness reveal that the two-step Simpson method turned into a block hybrid method (BHM) is the most accurate for the range of stiffness ratios given. The two-step Adams Moulton turned into a block hybrid method (BHM) is next.

So I am saying that stiffness is a mathematical problem, which is a real and serious matter for most traditional numerical methods. Since the stiff problem is important we cannot shy away from it rather some of us decided to "hold the bull by its horn".

4. DEVELOPMENT (PROGRESS) OF HIGHER ACCURATE NUMERICAL METHODS FOR STIFF PROBLEMS.

4.1 STIFF IVP OF THE FORM (1)

Since the publication of Dahlquist's barrier theorem in 1963, it seems like two rivers sprang up in two different directions in the search for higher accurate A-stable numerical methods for stiff problems.
The first one like the Benue River in Nigeria represents the one-step methods which have better stability properties and the second one like the Niger River in Nigeria which represents the multi-step methods with better accuracy properties and better efficiency but poor stability properties.

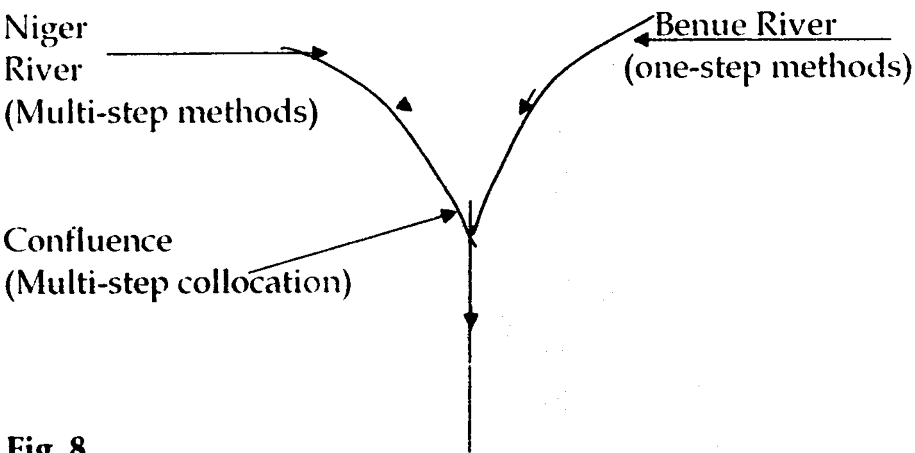


Fig. 8

The most important one-step methods are the implicit Runge Kutta (IRK) while the Adams and BDF in the Adams codes are the most important and most popular LMM used in the Adams code. Like Dahqluist, I am a multi-step person but listen to Shampine et al. (1980), who are RK authors, who said, "We have not been able to construct a variable order Runge-kutta code which can replace a good Adams code. If the equations are expensive to evaluate or if a lot of output is required, the Adams code is superior..."

ONE-STEP METHODS
CONTRIBUTION BY OTHER AUTHORS

The backward Euler (BE) method, which as we said earlier, historically is the first numerical method for solving stiff problems, was hoped upon, would lead to higher A -stable method with more points for derivative function evaluations in the formula. This thought yielded the one-step 3-stage Radau method of order five, an A-stable integrator that has proved to be successful (Hairer and Wanner (1996)).

A similar thought to involve the trapezoidal rule with an additional derivative function evaluation at $t_{i+1/2}$, mid-point between t_i and t_{i+1}, gave the A-stable method of order four by Gragg-Steller (1964). The most accurate class in this direction is the Lobatto method. The question that I keep asking myself is this. Why didn't the researchers pursue Gragg-Steller (1964) one step formula for higher steps k= 2,3, --- like it was done for the BE leading to the BDF (Curtiss and Hirschfelder (1952)). On the other hand, the Trapezoidal rule at the Gauss points x_{i+u} and x_{i+v} leads to the Gauss methods of optimal order p = 2s, s = 1,2, They are A-stable one-step methods. Unfortunately, this accurate class did not meet the L-stability and strong - stability properties of some one-step methods and so are not as successful.

On the whole, the flow of IRK methods through the past forty years has been impressive and intensive, led mainly by Burrage and Butcher (1980), Butcher (1987) and many others. It will continue to flow elegantly until it reaches a confluence with multi-step RK methods. Jackiewicz and Tracogna (1995) have already developed the two-step Runge-Kutta (TSRK) method with new improvements that recently appeared in Chollom and Jackiewicz (2003), (2004). They are members of the General Linear Methods (GLM) of Burrage and

Butcher (1980), where they said," . . . methods sufficiently general as to include linear multistep and one-step Runge-kutta methods as special cases . . . ". At this stage Hairer and Wanner have this summary to give, "In a remarkably short period (1964 - 1966) many independent papers appeared which tried to generalize either one - step Runge-kutta methods in the direction of multi-step or multistep methods in the direction of one - step Runge-kutta. The motivation was either to make the advantages of multistep accessible to Runge-kutta methods or to "break the Dahlquist barrier" by modifying the multistep formulas. Gragg and Stetter introduced "Generalized multistep methods" in (1964), "modified multistep methods" by Butcher (1964) and in the same year there appeared the work of Gear (1965) on "hybrid methods". A year later Byrne and Lambert (1966) published their work on "pseudo Runge-kutta methods". All these methods fell into the class of "general linear methods".

According to Butcher (1984), he said," following the advise of Aristotle, we look for the greatest good as a mean between extremes." The single hybrid methods (SHM) (Gear (1965), Lambert (1973) and Butcher (1964)) were developed from this thought of Professor Butcher.

MULTI- STEP METHODS

RESEARCH EFFORTS BY OTHER AUTHORS

Following the account given in Hairer and Wanner 1996, the early development of multi-step methods has been directed towards the improvement of the BDF methods (see Enwright (1974) for the Embedded BDF (EBDF) and Cash (1981, 1983) for the modified EBDF (MEBDF)). The emergence of multi-step collocation MC of Lie and Norsett (1989) has led to the new class of k-step 3-stage Radau methods of order p=k+2s, (s+1 stages where s=2 is the number of collocation points at Gauss points). The first two members k=1 and k=2 give A-stable methods. The higher step numbers $k \geq 3$ lead to A (α)- stable ($\alpha < 90^\circ$) methods. The use of an additional collocation point at t_{i+k} is because Radau methods are known to have the strong (s) stability property when k =1, Prothero and Robinson (1980)). The two-step three stage Radau method of order p=6 has now occupied "the top of the known accurate A-stable multi-step practical methods".

CONTRIBUTIONS BY ONUMANYI'S FORMULA

(a) The continuous output formula (4a) and (4b) or (5) is evaluated for the direct solution of the boundary value problem particularly the multistep methods (Adams-Moulton and the BDF). The idea of evaluation of formula (5) were studied simultaneously by Sirisena (1997) and Jator (1997) in their PhD works which developed into a large experimental work by Onumanyi, Sirisena and Jator (1999) using the block Adams Moulton (CAMM) method for a direct solution to non-stiff BVP. For stiff IVP, the chapter four of Sirisena's PhD thesis started with the block BDF and its hybrid methods for stiff initial value problems by considering the evaluation of k discrete multi - finite difference schemes from the continuous BDF methods, with close accuracy and stability properties by all the derived members. Yusuph (2003)'s p.hD thesis has shown that CAMM (1999) can be converted to Mutli - Step Implicit Runge - Kutta methods of order

$p = k + 1$ for $k = 1,2,3,4$; $p \leq 5$ are A - Stable Methods.

(b) The formula (5) gives the two-step three-stage Radau class mentioned above as special cases.

(c) We have found a new class of A-stable Adams Moulton Block Hybrid (AMBH) k-step methods of order p= k+2, k= 2,3, 4, 5 from the formula (5). We have not yet investigated the case k>5, (See Chollom and Onumanyi 2004).

(d) The formula (5) gives the A-stable Simpson Block Hybrid (SBH) two-step method of order p= 4.
The Fig. 9 below is shown to illustrate these developments pictorially.

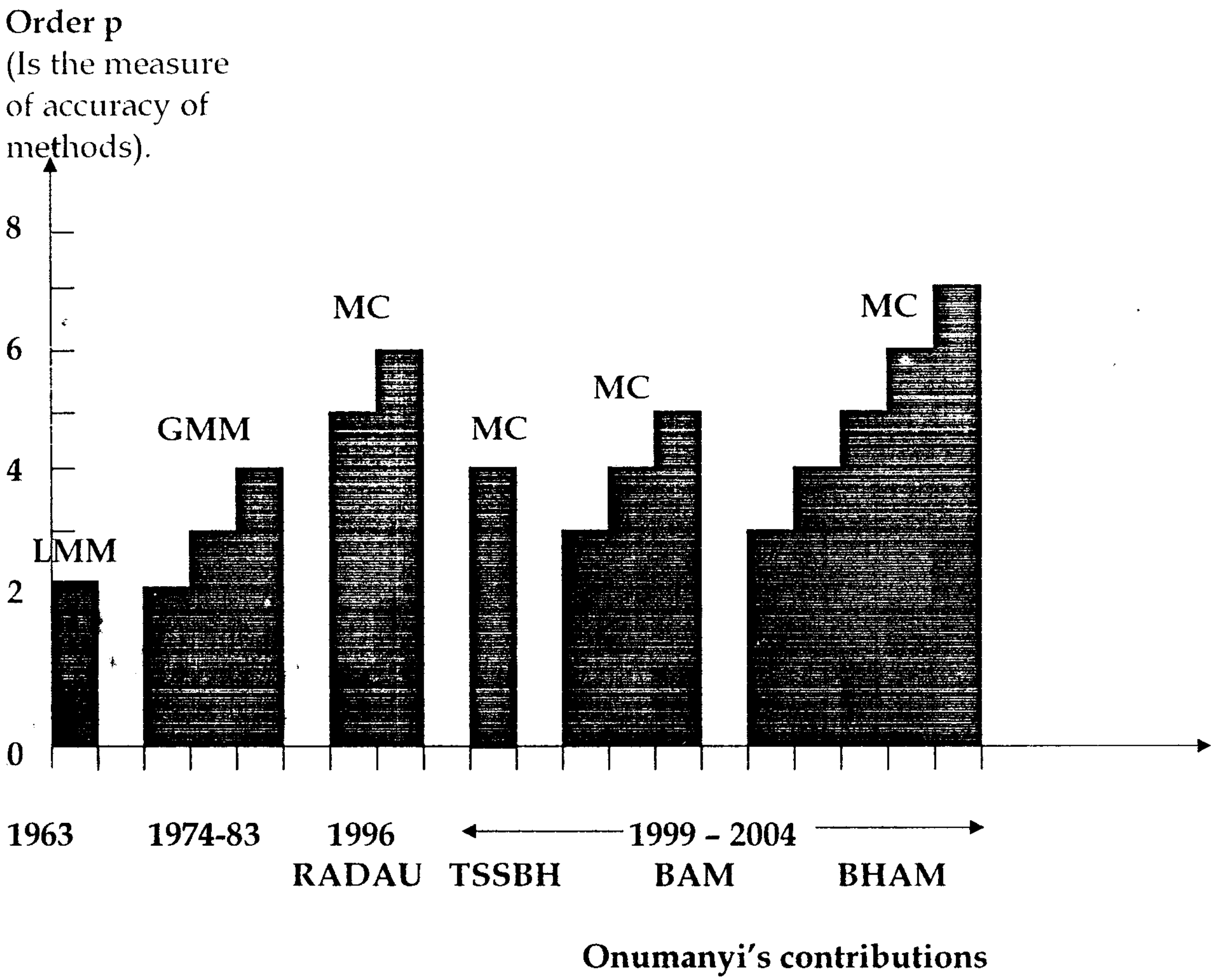


Fig 9. The graph shows progress in the development of accurate A-stable multi-step methods for stiff problems for the past forty years.

4.2 STIFF AND SINGULARLY PERTURBED PROBLEMS (BVP)

In the period of 1981-1991, I was interested in the numerical solutions of singular perturbation problems (SPP), which form a special class of problems containing a parameter ($\varepsilon>0$). When this parameter is small, the corresponding ODE is stiff.

My contribution in this problem was based on a combination of the quasi-linearization technique with three different polynomial numerical methods. They are:

(i) Segmented - Adaptive formulation of the Tau-method (Lanczos Tau method) - see Onumanyi and Ortiz (1984). A paper, which covered 15 pages of American mathematical society's leading and oldest Journal of numerical analysis. You carry your results to them when you convince yourself that you have really got something to show (mathematics of computation) (1984); vol.43, pp.167, 189-203.

(ii) Collocation Approximation combined with exponential fitting for nonlinear ODEs that are linearized for iteration.

(iii) Error Estimation in the Lanczos Chebyshev Reduction LCR method and application. The following three papers are worth mentioning.

(a) Adeniyi and Onumanyi (1991), error estimation in the numerical solution of ODEs with the tau method which appeared in Computers and Mathematics with Applications vol.21, no.9, pages 19-27 (9 pages) published in Great Britain by Pergamon Press, PLC.

(b) Ibiejugba, Odekunle and Onumanyi (1992), A computational Implementation of an error estimation of the Lanczos - Chebyshev Reduction (LCR) method for linear BVPs, which appeared in Journal of the NMS vol.13, pages 37 -51 (14pages)

(c) Odekunle, Ibiejugba and Onumanyi have been commended a couple of months ago by a letter from the AMS Abstract Review concerning their joint paper entitled as follows, A posteriori Error Estimator for Lanczos -Chebyshev Reduction Method.

5. APPLICATIONS

We are now about to land and for the purpose of a smooth landing, in this section, I have chosen some few easy to understand real life applications of mathematical problems/results. The first is for the young, which is not about stiff problem but about the Nigerian society. Courtesy of National Mathematical Center weekly newsletter called mathematical tit-bits edited by Professor L.O Adetula (2004).

1. An SS1 further mathematics student was traveling from one town to another and he was carrying 638 pebbles of stones with him. At every security check point along the road he threw away half of the pebbles through the door of the car and an additional pebble as well. On arrival at his destination he found he had 3 pebbles left. How many security checkpoints were along the road?

SOLUTION

Let n be the number of check points on the road and let p = 638 the number of pebbles he started with.

At the first checkpoint he had $(p/2 + 1)$ pebbles thrown away.

At the second check point it can easily be shown that he threw away $\frac{1}{2}(p/2 + 1)$.

At the third checkpoint again it can be shown that he threw away $\frac{1}{4}(p/2 + 1)$ and so on to the end of his journey.

You will then see that the ratios between the first and second checkpoints is $\frac{1}{2}(p/2 + 1)/ (p/2 + 1) = \frac{1}{2}$ and between the second and third checkpoints is $\frac{1}{4}(p/2 + 1) / \frac{1}{2}(p/2 + 1) = \frac{1}{2}$.

Let us assume that this ratio ½ is common to the end of his journey. Then, by the formula for the sum of a G.P,

$$p = 3 + (p/2 + 1)(1-(1/2)^n / 1 - \tfrac{1}{2}), \; p = 638.$$

Hence by working out these last equations using the log function of a scientific calculator you will get the answer n= 7.

What lessons have you learnt from this problem solving?

(a) It can be applied for any number n.

(b) P can be replaced by M or T.

(c) Do not throw away M or T at the security checkpoints. It is wrong; it is a crime if you do so (anti-corruption crusade).

(d) No security officer at the check points should force the road users for M or T because now there is a formula to expose you to the inspector General of Police once we determine the value n.

(e) The common ratio has been very crucial in the problem solving so is the stiffness ratio we talked about in the lecture for stiff problems.

In using the scientific calculator for obtaining the value n =7, the SS1 student never bothered how the answer came about but the answer was his interest.

The next example is about the mathematical function approximation embedded in an electronic machine (calculator or computer).

2. An unaware user of a scientific calculator or a digital computer does not realize that the pressing of a button of sine (x) function of his instrument triggers an approximation of the form

Sin (x) = a finite series of Chebyshev polynomials

not power series. The use of Chebyshev polynomials has been settled as the best min-max approximation in the1950 's.

So the next time you are using a scientific calculator or the computer, remember that we the numerical scientists made your approximate value most accurate to what you really want without the use of tables of functions anymore.

3. THE APPLICATION OF CARISLOTS SOFTWARE

The onshore / offshore Dichotomy Abrogation Law 2004 empowered the National boundary Commission of Nigeria to extend the interstate boundaries of Nigeria's Lithoral states to the 200-meter isobaths adjacent to them into the ocean as well. The methodology used according to the Director General of the Commission (2004) is the applicable CARISLOTS software that uses the equidistant Principle to share the areas adjacent to the Lithoral states. Complex problems of curved boundaries are involved in the software, which is tested, equitable and available for any users in the world.

STIFF PROBLEMS WITH APPLICATION TO REAL LIFE.

4. Both the young and the old are often involved in climbing very steep hills like the fig.4 shown earlier of the Adamawa hills and very steep staircases. While it is easy for the young, the old will soon experience panting at the top of the climbing.

What we have described is basically a distance (x) – time (t) problem involving velocity, which is a linear stiff ODE.

The old and those that are hypertensive should avoid these stiff exercises as they can seriously stress the heart, which can cause heart attack or heart failure.

5. Text-book, classroom and research Laboratory ODEs.

(i) A commonly used non-linear ODE is the Van der Pol's equation (see Chollom and Jackiewicz (2003), Ibiejugba and Onumanyi (1986), Oyelami, Ale and Onumanyi (2001), e.t.c)

The equation first appeared in 1935 by Van der pol in the study of electrical circuit oscillations. It has appeared in many other areas like Reaction-Diffusion Chemical Processes, Stability Problems of Structures, epidemiology that studies the control of spread of diseases and mathematical modeling of biological and life sciences and optimal control.

We considered the solution of the van der pol's two-point boundary value problem and applied it to a rapidly growing economy to reach the optimum growth (see Ibiejugba and Onumanyi (1986). The non-linear Van der Pol's equation is used as the non-linear constraint to the control problem. The fig.10 below shows a resemblance of a stiff characteristic.

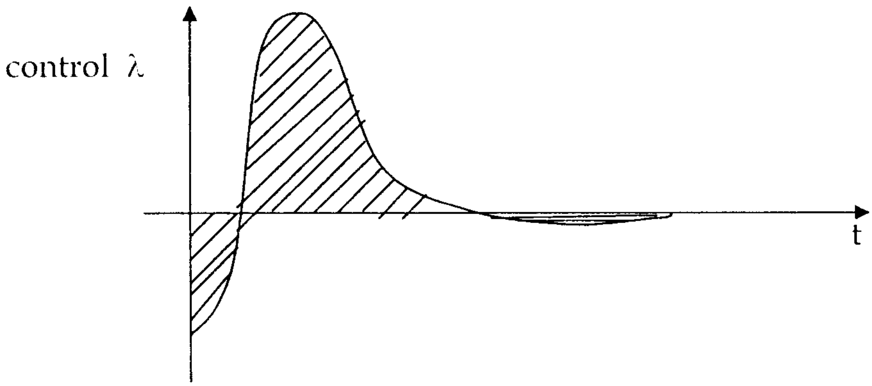


Fig 10

(ii) Ockendon (1980), who was assembling for Oxford Industry Problems several ODE arising from PDE models said, "text-book ODE problems rarely arise naturally in models of industrial processes. Instead, ODEs with unconventional non-linearities and boundary conditions often occur both naturally and from a study of PDE models. In such circumstances, numerical experimentation can be very fruitful in illuminating the whole mathematical model."

(iii) Let me add here that on the other hand these ODE problems are increasingly being used in the fast-growing field of Mathematical Biology (see a recent text-book "Essential Mathematical Biology" by N.F Britton, University of Bath, UK).

I conjecture that stiff problems will arise here, to bring about a cure to HIV/AIDS dreaded disease and similar STD. In this direction I refer you to recent research efforts by Kimbir and Aboiyar (2003), which they titled Mathematical model for the prevention of HIV/AIDS in a varying population."

Dr.Kimbir's idea is that it will be possible to eradicate the infection in finite time provided certain conditions hold. A typical graph resembles stiff characteristics as I pointed to him recently (see fig.11 below). A large parameter in the ODE can mean a high prevalence which is required to be reduced to zero at very short time is stiff problem.

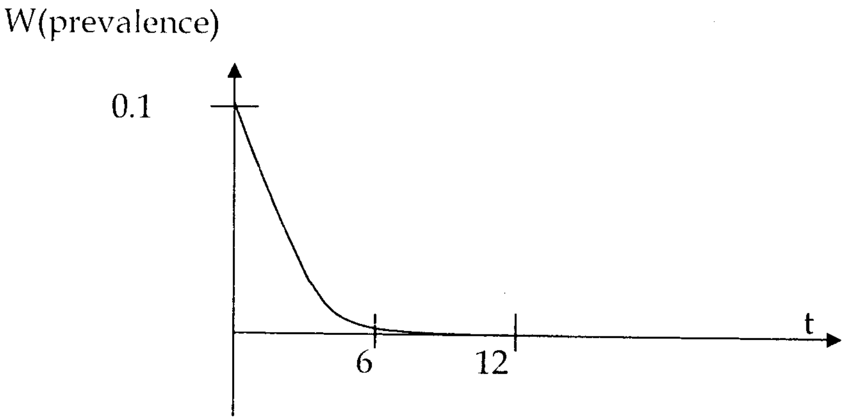


Fig 11

An attempt to study the spread of HIV/AIDS using B-transform (possessing B-stability property) with a numerical method approach has been made by Oyelami, Ale and Onumanyi (2001).
This is an area for future research.

(iv) The path of a space vehicle and war missiles need adjustments constantly to control it. The path is expected to be quite smooth, but very rapid corrections can be made in the course if any deviation from the programmed flight path is detected. Thousands of such equations of the form (1) are involved and quite often the Implicit Runge-kutta (IRK) methods are called upon to solve them. These are one-step methods, which have very good accuracy and stability properties (see Butcher (1987), Butcher (1964), Cash (1976), (1979)).

(v) A singular perturbation problem (SPP), which arises in chaotic theory, (Bifurcation of a solution into multiple ones) has been discussed as a stiff ODE in Kedem (1981), Onumanyi (1981), see fig.12.

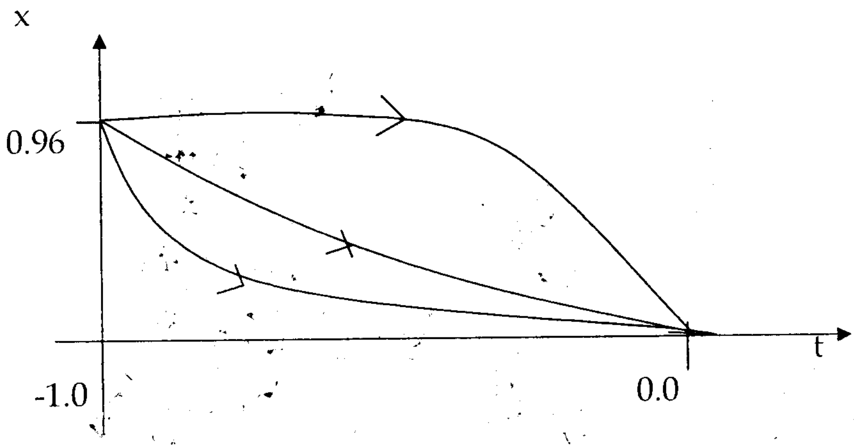


Fig 12

(vi) Another source of such problems is in the monitoring of chemical processes, since there can be a great diversity of time scales for the physical and chemical reaction systems. According to H.H Robertson (1966), "when the equations represent the behavior of a system containing a number of fast and slow reactions, a forward integration of these equations becomes difficult. The example of Robertson's (1966) has become very popular in numerical studies (Willoughby (1973)) and (Hairer and Wanner (1996)).

My future concern

How can I extend these simple ideas of success stories to PDEs? There are two options either directly with all the independent variables left continuous or reduce the PDEs to a system of ODEs by the LCR of Chen (1979) and Odekunle, ibiejugba, and Onumanyi (2002), the semi-discretization leading to MOL method leaving only one independent variable continuous and a new combination of LCR + MOL (Numerov discretization approach). This procedure is intended to yield an overall order four Numerov Method for the Laplace equation.

6. CONCLUSION

In the Adams codes the Adams -Moulton (AM) methods are used for solving "non-stiff" problems while in the same code because the AM methods have no acceptable stability property required for "stiff" problems the Backward Differentiation Formulae (BDF) though of lower accuracy than the AM are employed for the "stiff" problems.

It is the Adams codes that are still the most popular and superior to IRK codes, for accurate and efficient solution methods to stiff IVP. Following Hairer and Wanner 1996 account the multistep collocation methods by Lie and Norsett (1989) with super-convergence has helped to pave the way to the construction of k-step Radau 3-stage (s=3) methods which yielded A-stable methods of order $P \leq 6$ that is the most accurate A-stable multistep method to date.

Lie and Norsett showed that the BDF is a special case of the multistep collocation method. Their formulation was not general enough to include the Adams methods as special case. Onumanyi et.al. (1994, 1999) have shown that continuous finite difference methods based on a matrix inversion approach to the multistep collocation, exist in great number for a direct solution of the IVP and BVP of the form (1).

In this lecture we have seen that Adams methods and a class of their hybrid methods can be linked to the multi-step collocation methods. No one imagined hitherto that A-stable hybrid methods could be associated with the Adams family, and I believe these words from Professor Amali's inaugural lecture (1998) are appropriate here and I quote," it is very often not possible for one to keep on looking for the origin of everything one comes across every time. But it is sometimes essential for one to concentrate on looking for the origin of a thing, since such a search may lead one to see the thing in a different and perhaps deeper, broader, and clearer perspective. Penicillin was said to have been discovered in this way."

Finally, I said that between 1981-1990, I had done many other works on stiff and singularly perturbed problems (SSP) of different form from (1). You heard me say that one of such papers has received a recent commendation by the Americans.

To God be the glory.
I thank you all for coming, for listening and for your patience till now.
May God bless you all. Amen.

LIST OF REFERENCES

1. Adeniyi, R.B., Onumanyi,P. and Taiwo, O.A. (1989). A computational error estimate of the tau method for non-linear ordinary differential equations. Journal of Nigerian mathematical society, vol.9, pp. 21-32.
2. Adeniyi, R.B., and Onumanyi, P. (1991). Error estimation in the numerical solution of ordinary differential equations with the tau method. Journal of computers and mathematics with applications, vol.21, no.9, pp.19-27.
3. Afolabi, M.O., Ibiejugba, M.A, and Onumanyi,P. (1986). Numerical experiment with non-linear differential equations and applications to optimal control. The Nigerian journal of pure and applied sciences, vol.1, pp.26-38.
4. Aiken, R.C. ed. (1985). Stiff computation. Oxford, university press.
5. Amali, S.O.O. (1998). The seventh Inaugural lecture of the University of Jos. The Amalian two theories on cultural creativity and change.
6. Awoyemi, D.O. (1992). On some continuous linear multi-step methods for initial value problems. PhD thesis, Department of Mathematics, university of Ilorin, Ilorin. Nigeria (unpublished).
7. Burrage, K. and Butcher, J.C. (1980). Non-linear stability of a general class of differential equations methods. BIT, vol.20, pp.185-203.
8. Butcher, J.C. (1964). Implicit Runge-Kutta processes. Math. comput. vol.18, pp.50-64.
9. Butcher, J.C. (1984). An application of the Runge-kutta space. BIT, vol.24, pp. 425-440.
10. Butcher, J.C. (1987). Linear and non-linear stability for general linear methods. BIT, vol.27, pp.182-189.
11. Byrne, G.D. and Lambert, R.J. (1966). Pseudo-Runge-Kutta methods involving two points. J. assoc. comput. Mach., vol.13, pp.114-123.
12. Cash, J.R. (1976). Semi-implicit Runge-Kutta procedures with error estimates for the numerical integration of stiff systems of ordinary differential equations. JACM, vol.23, pp.455-460.

13. Cash, J.R. (1979). Diagonally implicit Runge-Kutta formulae with error estimates. J. Inst. Math. Applics. Vol.24, pp.293-301.
14. Chollom, J.P. and Jackiewicz, Z. (2003). Construction of two-step Runge-Kutta methods with large regions of absolute stability. J. Comput. and Applied Mathematics, vol.157, pp.125-137.
15. Chollom, J.P. and Jackiewicz, Z. (2004). (Submitted).
16. Chollom, J.P. and Onumanyi, P. (2004). High order A-stable General Linear Methods. Abacus, J. Math. Assoc. Nigeria. Vol. - - - ser. B. pp. - - -
17. Curtiss, C.F. and Hirschfelder, J. O. (1952). Integration of stiff equations. Proc. Nat. Acad. Sci. vol.38, pp. 235-243.
18. Dahlquist, G. (1993). A special stability problem for linear multi-step methods. BIT. Vol.3, pp. 27-43.
19. Enright, W.H. (1974). Second derivatives multi-steps methods for stiff ordinary differential equations. SIAM J. Numer. Anal., vol.11, pp.321-331.
20. Gear, C.W. (1965). Hybrid methods for initial value problems in ordinary differential equations. SIAM J. Numer. Anal. ser.b, vol.2, pp.69-86.
21. Gragg, W.B. and Stetter, H.J. (1964). Generalized multi-step predictor corrector methods. J. ACM, vol.11, pp. 188-209.
22. Hairer, E , Norsett, S.P. and Wanner, G (1993), 2nd rev. ed. Vol. I. Non-stiff problems. Springer Verlag. Springer series in computational mathematics, vol.8.
23. Hairer, E. and Wanner, G. (1996). Solving ordinary differentials equations, vol II. Stiff and differential-algebraic problems. 2nd rev. ed. springer verlag. Springer series in computational mathematics vol.13.
24. Ibiejugba, M.A., Odekunle, R. and Onumanyi, P. (1992). A computational implementation of an error estimation of the Lanczos-Chebyshev method for linear boundary value problems. J. Nigerian mathematical society. Vol.13, pp.37-51.
25. Jackewicz, Z. and Tracogna, S. (1995). A general class of two-step Runge-Kutta methods for ordinary differential equations. SIAM J. Numer. Anal. vol.32, pp.1390-1427.

26. Jator,S.N. (1997). Continuous Adams-Moulton methods for the direct solution of non-stiff ordinary differential equations with error estimations. PhD thesis, Department of Mathematics, University of Ilorin, Ilorin, Nigeria (unpublished).
27. Kedem,G. (1981). A posteriori error bounds for two-point boundary value problems, SIAM J. Numer. Anal. 18, pp.431-448.
28. Kimbir, A.R. and Aboiyar,T. (2003). A mathematical model for the prevention of a HIV/AIDS in a varying population. J. Nig. Math. soc. Vol.22, pp. 43-55.
29. Lambert. J.D. (1973). Computational methods in ordinary differential equations. Willy and sons.
30. Lie, I. and Norsett, S.P. (1989). Super convergence for multi-step collocation. Math. of comput. Vol.52, pp.65-79.
31. Odekunle, M.R., Ibiejugba, M.A. and Onumanyi, P. (2002). A posteriori Error Estimator for Lanczos -Chebyshev Reduction Method. J. of comput. and Applied Mathematics..
32. Onumanyi, P. (1981). Numerical solution of boundary value problems with the tau method. PhD thesis of the university of London, Department of Mathematics, Imperial College of Science and Technology, London, S.W. 7. UK.
33. Onumanyi, P. and Ortiz, E.L. (1984). Numerical solution of stiff and singularly perturbed boundary value problems with a segmented-adaptive formulation of the tau method. Math. of comput. vol.43, pp. 189-203.
34. Onumanyi, P., Awoyemi, D.O., Jator,S.N. and Sirisena, U.W. (1994). New linear multi-step methods with continuous coefficients for first order IVPs. J. of Nigerian mathematical society, vol.13, pp. 37-51.
35. Onumanyi, P., Sirisena, U.W., and Jator, S.N. (1999). Continuous finite difference approximations for solving differential equations. J.of comput. Math. vol.72, no1, pp.15-27.
36. Onumanyi, P., Sirisena, U. W. and Adee, S. (2002). Some theoretical considerations of continuous linear multi-step methods for U(v) = f(x, u), v = 1,2. J. of Pure and Applied Sciences, vol.2, No.2, pp.1-5.

37. Oyelami, Ale, and Onumanyi, P. (2001).
38. Prothero, A. and Robinson, A. (1974). On the stability and accuracy of one-step for solving stiff systems of ordinary differential equations. Math. of comput. vol.28, pp. 145-162.
39. Robertson, H.H. (1966). The solution of a set of reaction rate equations. In J. Walsh ed. Numer. Anal., an introduction, Academ. Press. pp. 178-182.
40. Shampine, L.F., Gordon, M.K. and Wisniewski, J.A. (1980). Variable order Runge-Kutta codes. In computational techniques for ordinary differential equations. ed. by Gladwell, I. and sayers, D.K. Academic press, London.
41. Sirisena, U.W. (1997). A reformulation of the continuous general linear multi-step method by matrix inversion for the first order initial value problems. PhD thesis, Department of Mathematics, University of Ilorin, Ilorin, Nigeria. (Unpublished).
42. Van der Pol. (1935). Tchebysheff polynomials, Physica, pp.218-236.
43. Willoughby, R.A. ed.(1973). Stiff differential systems. Plenum.

BIOGRAPHY OF THE AUTHOR

Professor Peter Onumanyi was born in 1951 in Okene town of Kogi State of Nigeria. He attended the RCM primary school, Okene-South and had his secondary education at Government Secondary School, Dekina from where he obtained a division-one in the 1968 West African School Certificate Examination.

He proceeded to Titcombe College, Egbe, Kogi State for the Higher School Certificate course in 1969. Professor Onumanyi later attended the University of Ibadan, where he graduated in Mathematics in 1975 with the B.Sc. Hons. Degree, thereafter, he obtained the M.Sc. and Ph.D (Numerical Analysis) degrees from the Imperial College of Science and Technology, University of London in 1979 and 1981, respectively. He obtained his M.Sc. degree with "a mark of distinction" at the Imperial College, London.

Professor Onumanyi has taught and carried out research in Mathematics at the department of Mathematics, University of Ilorin, Ilorin, Kwara State of Nigeria (1982-1992), University of Jos, Jos, Plateau State of Nigeria (1992-2004). He has singly supervised eight (8) students that have obtained Ph.D degree in Numerical Analysis and over twenty-five (25) for the M.Sc. degree in Numerical Analysis.

The University of Jos appointed him a professor of Mathematics in 1992 and he is still very active in research and paper writing. He has forty (40) scientific publications in Journals across Africa, Europe, Japan and USA.

Professor Peter Onumanyi has assessed many of his academic colleagues' papers for professorship appointments. He has served and still serving as external examiner to many Nigerian Universities and Polytechnics.

He has served in many editorial boards of academic journals and mathematical abstracts reviews both in Nigeria and overseas. Among them are Mathematical Abstracts reviewers for Zentrablatt for Mathematic, West Germany (1983-1993), editor in chief of Abacus, Journal of Mathematical Association of Nigeria (MAN) (1989-1994), Member, Editorial board of Abacus (1994-2004).

Professor Peter Onumanyi has served on many important committees as chairman or member within and outside the University Communities of

University of Jos and University of Ilorin. He has served as the National President of the Mathematical Association of Nigeria (MAN) (1998-2000). He is a life member of the Association and awarded the highest honour of the Association Fellow of MAN (FMAN) in 2001. He has held the following positions in the University of Jos, Head of Department of Mathematics (1993-1995), Dean, Faculty of Natural Sciences (1995-1998), Chairman, Committee of Deans (1997-1998), elected representative of the congregation to the University Governing Council (1997-2000) and Deputy Vice-chancellor (Academic) (1998-2000), Professor Peter Onumanyi is married and has six children.

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S/N	NAME	TITLE	DATE	LECTURE SERIES
1.	Professor E. Isichei	Towards A History of Plateau State.	10th February, 1981	1
2.	Professor A.C. Ikeme	The End of A Myth: The Evolution of Cardiology in Africa.	21st January, 1983.	2
3.	Professor P.N. Lassa	The Sorry State of Mathematics Education in Nigeria	20th January, 1984.	3
4.	Professor G.O.M. Tasie.	The Vernacular Church and Nigerian Society.	2nd July, 1997.	4
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