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HIGH ORDER BLOCK IMPLICIT MULTI-STEP (HOBIM) METHODS FOR THE SOLUTION OF STIFF ORDINARY DIFFERENTIAL EQUATIONS

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Abstract: The search for higher order A-stable linear multi-step methods has been the interest of many numerical analyst and has been realized through either higher derivatives of the solution or by inserting additional off step points,supper future points and the likes.These methods are suitable for the solution of stiff differential equations which exhibit characteristics that place severe restriction on the choice of step size. It becomes necessary that only methods with large regions of absolute stability remain suitable for such equations. In this paper, high order block implicit multi-step methods of the hybrid form up to order twelve have been constructed using the multi-step collocation approach by inserting one or more off step points in the multi-step method. The accuracy and stability properties of the new methods are investigated and are shown to yield A- stable methods, a property desirable of methods suitable for the solution of stiff ODEs. The new High Order Block Implicit Multistep methods used as block integrators are tested on stiff differential systems and the results reveal that the new methods are efficient and compete favorably with the state of the art Matlab ode23 code.

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1. Introduction

Stiff differential equations have been known to cause singular computational difficulties because of the severe restriction on the step size used. To solve the stiff ODE (1), various authors have made several attempts and came up with various methods of solution.

$$y' = f(x, y), \quad y(0) = 0. \tag{1}$$

The traditional approach has been the Adams code developed by Shampine [28] where the Adams Moulton formula was used as predictor and Adams Bashfort formula as corrector. Though these yielded a very successful combination but have a draw back because of its requirements for starting values which could lead to growing numerical errors and corrupting further approximations Mayers and Suli[24]. To resolve the issue of starting value, Onimanyi etal[25],[26] proposed the block Linear Multistep methods based on the multi-step collocation approach of Lie and Norset [23] and the self-starting methods of Cash [6]. These methods were developed through the continuous formulation of the linear k-step methods which provided sufficient number of simultaneous discrete methods used as single integrators. However most of these methods were not able to handle stiff ODEs due to stability issues. A popular approach that worked for this class of methods was that by Gear [17],Ascher and Petzold [2] who developed the Backward Differentiation formula of step number k and order k with distinguish features where f is evaluated at the current step $[x_n, y_n]$ only. The k-step Adams Moulton methods have failed to perform successfully on stiff equations . Researchers then resorted to implicit Runge-Kutta (IRK) methods, the Backward Differentiation Formulae (BDF) and recently the collocation methods (Hairer and Wanner [19] and Enright [13].These methods have been useful in handling stiff equations due to their better stability properties. To satisfy the A-stability property,the following researchers, Enright [13], Cash [7], [8], [9], Lie and Norsett [23], Chollom [11],Okuonghae and Ikhile [27],Akinfenwa,et al [3] and Ezzeddine and Hojjati [14] developed numerical methods that are A-stable and very suitable for the solution of stiff equations. In this paper, we pursue the hybrid block approach of Chollom and Onumanyi [12] were they constructed block hybrid Adams Moulton methods for $1 \leq K \leq 5$ which produced methods that are shown to possess better stability properties than single integrators and suitable for stiff ODE's.The paper constructed high order block implicit multi-step (HOBIM) methods for $6 \leq K \leq 10$ to advance the integration forward.. The rest of the paper is divided as follows: The derivation of the new methods is done in Section 2, the convergence analysis is in Section

3, Numerical experiment to test the efficiency of the new methods is done in Section 4 and the concluding remarks in Section 5

2. Derivation of the HOBIM Methods

The high order block implicit multi-step (HOBIM) methods are derived from a class of the multi-step collocation methods with continuous coefficients of the Adams class.These methods in their continuous form are expressed as:

$$y(x) = \sum_{j=0}^{k} \psi_j y_{n+j} - h \sum_{j=0}^{k} \beta_j f_{n+j} \tag{2}$$

for k the step number $k > 0$,h a constant step size given by $h = x_{n+r} - x_n, r = 1, 2, ..., k$.The approximation (2) is formulated into the generalization

$$\overline{y}(x) = \sum_{j=0}^{t-1} \psi_j y_{n+j} = h \sum_{j=0}^{s-1} \beta_j f(\overline{x}, \overline{y}(\overline{x})) \tag{3}$$

And is defined over the interval $x \in (x_{n+k-1}, x_{n+k})$, where

$$\Psi_j(x) = \sum_{j=0}^{r+m-1} \alpha_{j,j+1} x^j h\beta_j(x) = h \sum_{j=0}^{r+m-1} \beta_{j,j+1} x^j \tag{4}$$

are the continuous coefficients to be determined satisfying the following interpolation and collocation conditions.

$$\begin{array}{l} y(x_{n+j}) = y_{n+j}, j \in (0, 1, \ldots, r-1) \\ y^{'}(x_j) = f_{n+j}, j \in (0, 1, \ldots, m-1) \end{array} \tag{5}$$

where $f_{n+j} = f(x_{n+j}, y_{n+j})$. The interpolation and collocation conditions being $r - 1$ and $m - 1$ respectively .From equations (4) and (5) we have the following imposed conditions:

$$\begin{array}{l} \alpha_j x_{n+j} = \delta_{ij}, j \in (0, 1, \ldots, r-1), i \in (0, 1, \ldots, r-1) \\ h\beta_j x_{n+j} = 0, j \in (0, 1, \ldots, r-1), i \in (0, 1, \ldots, r-1) \\ \alpha_j^{'} x_j = 0, j \in (0, 1, \ldots, r-1), i \in (0, 1, \ldots, r-1) \\ h\beta_j^{'} x_{n+j} = \delta_{ij}, j \in (0, 1, \ldots, m-1), i \in (0, 1, \ldots, r-1) \end{array} \tag{6}$$

Expressing (6) in matrix form yields the equation

$$DC = I \tag{7}$$

I being an identity matrix of order $r \times m$ and the matrix D given by

$$D = \begin{pmatrix} 1 & x_n & x_n^2 & \dots & x_n^t & \dots & x_n^{t+s-1} \\ 1 & x_{n+1} & x_{n+1}^2 & \dots & x_{n+1}^t & \dots & x_{n+1}^{t+s-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_{n+k-1} & x_{n+k-1}^2 & \dots & x_{n+k-1}^t & \dots & x_{n+k-1}^{t+s-1} \\ 0 & 1 & 2x_n & \dots & (t)x_n & \dots & (t+s-1)x_n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2x_{s-1} & \dots & (t)x_{s-1}^{t-1} & \dots & (t+s-1)x_{s-1}^{(t+s-2)} \end{pmatrix} \quad (8)$$

is a non-singular matrix of dimension (s+t) $\times$ (s+t).The matrix C given in (9) and is of

$$C = \begin{pmatrix} \alpha_{01} & \alpha_{11} & \dots & \alpha_{t-1,1} & h\beta_{01} & \dots & h\beta_{s-1,1} \\ \alpha_{02} & \alpha_{12} & \dots & \alpha_{t-1,2} & h\beta_{02} & \dots & h\beta_{s-1,2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{0,t+s} & \alpha_{1,t+s} & \dots & \alpha_{t-1,t+s} & h\beta_{0,t+s} & \dots & h\beta_{s-1,t+s} \end{pmatrix} \quad (9)$$

dimension (s+t) $\times$ (s+t) defined by $C = D^{-1} = C_{ij}, i, j = 1,\dots, s+t-1$ The entries of the matrix C substituted into (4) produces the continuous coefficients of the method referred to as the continuous formulation of the Adams Moulton Class. The continuous interpolant evaluated at both grid and off grid points results in the methods discrete schemes used as block integrators.

2.1. Derivation of HOBIM $k = 6$

Considering $k = 6, x \in [x_n, x_{n+6}]$ .The continuous form of the HOBIM method $k = 6$ is given by

$$y(x) = \sum_{j=0}^{k} \psi_j(x) y_{n+j} + h \sum_{j=0}^{k} \beta_j(x) f_{n+j} + \beta_j(\mu), \mu = \frac{11}{2} \quad (10)$$

Evaluating the matrix (8) with $s = 7, t = 2$ produces the matrix of the method D in (11)

$$D = \begin{pmatrix} 1 & x_{n+5} & x_{n+5}^2 & \dots & x_{n+5}^8 \\ 0 & 1 & 2x_n & \dots & 8x_n^7 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2x_{n+\mu} & \dots & 8x_{n+\mu}^7 \\ 0 & 1 & 2x_{n+6} & \dots & 8x_{n+6}^7 \end{pmatrix} \quad (11)$$

Inverting (11) in a Maple environment yields the elements of the matrix C in (9).From the elements of C we obtain the continuous coefficients of the HOBIM method for $k = 6$ in (12).

$$
\begin{aligned}
&\alpha_5(x) = 1\\
&h\beta_0(x) = \frac{579\zeta^2}{440h} + \frac{1337\zeta^3}{1485h^2} - \frac{11333\zeta^4}{31680h^3} + \zeta + \frac{679\zeta^5}{7920h^4} - \frac{581\zeta^6}{47520h^5} - \frac{295}{1008h} - \frac{53\zeta^7}{55440h^5} - \frac{\zeta^8}{31680h^7}\\
&h\beta_1(x) = \frac{11\zeta^2}{3h} - \frac{359\zeta^3}{90h^2} + \frac{2117\zeta^4}{1080h^3} - \frac{191\zeta^5}{360h^4} + \frac{53\zeta^6}{648h^5} - \frac{28025}{18144h} - \frac{17\zeta^7}{2520h^5} - \frac{\zeta^8}{4320h^7}\\
&h\beta_2(x) = -\frac{165\zeta^2}{28h} + \frac{67\zeta^3}{8h^2} - \frac{925\zeta^4}{192h^3} + \frac{347\zeta^5}{240h^4} - \frac{23\zeta^6}{96h^5} - \frac{125}{672h} - \frac{\zeta^7}{48h^5} - \frac{\zeta^8}{1344h^7}\\
&h\beta_3(x) = \frac{22\zeta^2}{3h} - \frac{1517\zeta^3}{135h^2} + \frac{1277\zeta^4}{180h^3} - \frac{83\zeta^5}{36h^4} + \frac{11\zeta^6}{27h^5} - \frac{1975}{1008h} - \frac{47\zeta^7}{1260h^5} - \frac{\zeta^8}{720h^7}\\
&h\beta_4(x) = \frac{55\zeta^2}{8h} - \frac{131\zeta^3}{12h^2} - \frac{4169\zeta^4}{576h^3} + \frac{199\zeta^5}{80h^4} - \frac{401\zeta^6}{864h^5} - \frac{125}{756h} + \frac{5\zeta^7}{112h^5} - \frac{\zeta^8}{576h^7}\\
&h\beta_5(x) = \frac{33\zeta^2}{5h} - \frac{107\zeta^3}{10h^2} + \frac{877\zeta^4}{120h^3} - \frac{313\zeta^5}{120h^4} + \frac{61\zeta^6}{120h^5} - \frac{995}{972h} - \frac{43\zeta^7}{840h^5} + \frac{\zeta^8}{489h^7}\\
&h\beta_{\frac{11}{2}}(x) = \frac{55\zeta^2}{8h} - \frac{131\zeta^3}{12h^2} - \frac{4169\zeta^4}{576h^3} + \frac{199\zeta^5}{80h^4} - \frac{401\zeta^6}{864h^5} - \frac{125}{756h} + \frac{5\zeta^7}{112h^5} - \frac{\zeta^8}{576h^7}\\
&h\beta_6(x) = \frac{55\zeta^2}{8h} - \frac{131\zeta^3}{12h^2} - \frac{4169\zeta^4}{576h^3} + \frac{199\zeta^5}{80h^4} - \frac{401\zeta^6}{864h^5} - \frac{125}{756h} + \frac{5\zeta^7}{112h^5} - \frac{\zeta^8}{576h^7}
\end{aligned}
\tag{12}
$$

Substituting (12) into (10) with $\mu = \frac{11}{2}$,$\zeta = x - x_n$ produces the continuous form of the multistep collocation method for K= 6 as:

$$
\begin{aligned}
\bar{y}(x) = {}& y_{n+5} + [-\frac{579}{440h}\zeta^2 + \frac{1337}{1485h^2}\zeta^3 - \frac{1133}{31680h^3}\zeta^4 + \zeta + \frac{679}{7920h^4}\zeta^5 - \frac{581}{47520h^5}\zeta^6 - \frac{295}{1008h} - \frac{53}{55440h^6}\zeta^7\\
&- \frac{1}{31680h^7}\zeta^8]f_n + [\frac{11}{3h}\zeta^2 - \frac{359}{90h^2}\zeta^3 + \frac{2117}{1080h^3}\zeta^4 - \frac{191}{360h^4}\zeta^5 + \frac{53}{648h^5}\zeta^6 - \frac{28025}{18144h} - \frac{17}{2520h^6}\zeta^7\\
&- \frac{1}{4320h^7}\zeta^8]f_n + 1 + [-\frac{165}{28h}\zeta + \frac{67}{8h^2}\zeta^3 - \frac{925}{192h^3}\zeta^4 + \frac{347}{240h^4}\zeta^{\frac{23}{96h^5}}\zeta^6 - \frac{125}{672h} + \frac{1}{48h^6}\zeta^7 - \frac{1}{1344h^7}\zeta^8]f_{n+2}\\
&+ [\frac{22}{3h}\zeta^2 - \frac{1517}{135h^2}\zeta^3 + \frac{1277}{180h^3}\zeta^4 - \frac{83}{36h^4}\zeta^5 + \frac{11}{27h^5}\zeta^6 - \frac{1975}{1008h} - \frac{47}{1260h^6}\zeta^7 + \frac{1}{720h^7}\zeta^8]f_{n+3}\\
&+ [-\frac{55}{8h}\zeta^2 + \frac{131}{12h^2}\zeta^3 - \frac{4169}{576h^3}\zeta^4 + \frac{199}{80h^4}\zeta^5 - \frac{401}{864h^5}\zeta^6 - \frac{125}{756h} + \frac{5}{112h^6}\zeta^7 - \frac{1}{576h^7}\zeta^8]f_{n+4} + [\frac{33}{5h}\zeta^2\\
&- \frac{107}{10h^2}\zeta^3 + \frac{877}{120h^3}\zeta^4 - \frac{313}{120h^4}\zeta^5 + \frac{61}{120h^5}\zeta^6 - \frac{995}{972h} - \frac{43}{840h^6}\zeta^7 + \frac{1}{489h^7}\zeta^8]f_{n+5} + [\frac{11}{12h}\zeta^2 - \frac{1627}{1080h^2}\zeta^3\\
&+ \frac{3023}{2880h^3}\zeta^4 - \frac{227}{720h^4}\zeta^5 + \frac{67}{864h^5}\zeta^6 - \frac{275}{2016h} - \frac{41}{5040h^6}\zeta^7 + \frac{1}{2880h^7}\zeta^8]f_{n+\frac{11}{2}} + [\frac{33}{5h}\zeta^2 - \frac{107}{10h^2}\zeta^3\\
&+ \frac{877}{120h^3}\zeta^4 - \frac{313}{120h^4}\zeta^5 + \frac{61}{120h^5}\zeta^6 - \frac{995}{756h} - \frac{43}{840h^6}\zeta^7 + \frac{1}{480h^7}\zeta^8]f_{n+6}
\end{aligned}
\tag{13}
$$

Table 1: Coefficients of HOBIM $k = 6$

k	1	2	3	4	5	$\frac{11}{2}$	6
α_0	0	0	0	0	1	0	0
α_5	1	1	1	1	0	1	1
β_0	$\frac{8}{945}$	-$\frac{78}{36960}$	$\frac{45}{12474}$	$-\frac{262}{332640}$	$\frac{5300}{18144}$	-$\frac{2335}{15482880}$	$\frac{18}{90720}$
β_1	-$\frac{342}{945}$	$\frac{9789}{36960}$	-$\frac{550}{12474}$	$\frac{2475}{332640}$	$\frac{28025}{18144}$	$\frac{21150}{15482880}$	-$\frac{157}{90720}$
β_2	-$\frac{1224}{945}$	-$\frac{15939}{36960}$	$\frac{3861}{12474}$	-$\frac{11187}{332640}$	$\frac{3375}{18144}$	-$\frac{88893}{15482880}$	$\frac{621}{90720}$
β_3	-$\frac{6641}{945}$	-$\frac{41910}{36960}$	-$\frac{53064}{12474}$	$\frac{35354}{332640}$	$\frac{35550}{18144}$	$\frac{239732}{15482880}$	-$\frac{1494}{90720}$
β_4	-$\frac{1224}{945}$	-$\frac{35904}{36960}$	-$\frac{148929}{12474}$	-$\frac{190872}{332640}$	$\frac{300}{18144}$	-$\frac{540873}{15482880}$	$\frac{2496}{90720}$
β_5	-$\frac{342}{945}$	-$\frac{2366}{36960}$	-$\frac{9994}{12474}$	-$\frac{219285}{332640}$	$\frac{58735}{18144}$	$\frac{4566222}{15482880}$	$\frac{11043}{90720}$
$\beta_{\frac{11}{2}}$	0	$\frac{6656}{36960}$	$\frac{10240}{12474}$	$\frac{59904}{332640}$	-$\frac{12800}{18144}$	$\frac{3732480}{15482880}$	$\frac{64000}{90720}$
β_6	$\frac{8}{945}$	-$\frac{1023}{36960}$	-$\frac{1089}{12474}$	-$\frac{876}{332640}$	$\frac{2475}{18144}$	-$\frac{186043}{15482880}$	$\frac{14103}{90720}$

Evaluating (13) at$\zeta = 0, h, 2h, 3h, 4h, \frac{11}{2}$,and $6h$ yields the coefficients of the discrete HOBIM method for $k = 6$ used used as block integrator given in Table1.

2.2. Derivation of HOBIM $k = 7$

Considering $k = 7, x \in [x_n, x_{n+7}]$ The continuous form of the method HOBIM k=7 is given by

$$y(x) = \sum_{j=0}^{k} \psi_j(x) y_{n+j} + h \sum_{j=0}^{k} \beta_j(x) f_{n+j} + \beta_j(\mu), \mu = \frac{13}{2} \tag{14}$$

Evaluating the matrix (8) with $s = 8, t = 2$ produces the matrix of the method D in (15)

$$D = \begin{pmatrix} 1 & x_{n+6} & x_{n+6}^2 & \dots & x_{n+6}^9 \\ 0 & 1 & 2x_n & \dots & 9x_n^8 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2x_{n+\mu} & \dots & 9x_{n+\mu}^8 \\ 0 & 1 & 2x_{n+7} & \dots & 9x_{n+7}^8 \end{pmatrix} \tag{15}$$

Inverting (15) in a Maple environment yields the elements of the matrix C .Substituting the elements of C into (4) produces the continuous coefficients of the method which are substituted into (14) to give the continuous form of the multistep collocation method (16).

$$
\begin{aligned}
y(x) = y_{n+6} &+ [-\frac{4999}{3640h}\zeta^2 + \frac{49213}{49140}h^2\zeta^3 - \frac{5441}{12480h^3}\zeta^4 + \zeta - \frac{929}{7800h^4}\zeta^5 - \frac{193}{9360h^5}\zeta^6 \\
&- \frac{2593}{9100h} - \frac{1}{455h^6}\zeta^7 - \frac{23}{17420}h^7\zeta^8 + \frac{1}{294840}h^8\zeta^9]f_n + [-\frac{91}{22h}\zeta^2 - \frac{20}{60}h^2\zeta^3 \\
&+ \frac{10301}{3960h^3}\zeta^4 + \zeta - \frac{31853}{39600}h^4\zeta^5 + \frac{1433}{9504h^5}\zeta^6 - \frac{3096}{1925h} - \frac{941}{5440h^6}\zeta^7 \\
&+ \frac{67}{63360h^7}\zeta^8 - \frac{1}{3540}h^8\zeta^9]f_{n+1} + [-\frac{91}{12h}\zeta^2 + \frac{1361}{120h^2}\zeta^3 - \frac{12325}{1728h^3}\zeta^4 \\
&+ \frac{13159}{5400}h^4\zeta^5 - \frac{635}{1296h^5}\zeta^6 + \frac{33}{70h} + \frac{439}{7560h^6}\zeta^7 - \frac{13}{3456}h^7\zeta^8 + \frac{1}{9720}h^8\zeta^9]f_{n+2} \\
&+ [\frac{65}{6h}\zeta^2 - \frac{13177}{756h^2}\zeta^3 + \frac{143}{12h^3}\zeta^4 - \frac{1051}{240h}{}^4\zeta^5 + \frac{269}{288h^5}\zeta^6 - \frac{86}{35h} \\
&- \frac{13}{112h^6}\zeta^7 + \frac{1}{128h^7}\zeta^8 - \frac{1}{4536}h^8\zeta^9]f_{n+3} + [\frac{91}{8h}\zeta^2 + \frac{284}{15h^2}\zeta^3 - \frac{38989}{2880h^3}\zeta^4 \\
&+ \frac{9409}{1800h}{}^4\zeta^5 - \frac{505}{432h^5}\zeta^6 + \frac{369}{700h} + \frac{191}{1260h^6}\zeta^7 - \frac{61}{5760h^7}\zeta^8 + \frac{1}{3240}h^8\zeta^9]f_{n+4} \\
&+ [\frac{91}{10h}\zeta^2 - \frac{309}{30h^2}\zeta^3 + \frac{4087}{360h^3}\zeta^4 - \frac{16313}{3600h}{}^4\zeta^5 + \frac{454}{4320h^5}\zeta^6 - \frac{396}{175h} \\
&- \frac{713}{5040h^6}\zeta^7 + \frac{99}{5760h^7}\zeta^8 - \frac{1}{3240}h^8\zeta^9]f_{n+5} + [\frac{91}{12h}\zeta^2 + \frac{14087}{1080h^2}\zeta^3 \\
&- \frac{9371}{960h^3}\zeta^4 + \frac{2393}{600h}{}^4\zeta^5 - \frac{137}{144h^5}\zeta^6 - \frac{299}{700h} + \frac{37}{280h^6}\zeta^7 - \frac{19}{1920h^7}\zeta^8 \\
&+ \frac{1}{3340}h^8\zeta^9]f_{n+6} + [\frac{2048}{429h}\zeta^2 - \frac{11264}{1365h^2}\zeta^3 + \frac{120064}{19305h^3}\zeta^4 - \frac{247552}{96525h}{}^4\zeta^5 \\
&+ \frac{7168}{11583h^5}\zeta^6 - \frac{12288}{25025h} - \frac{11776}{135135h^6}\zeta^7 + \frac{128}{19305h^7}\zeta^8 - \frac{256}{1216215}h^8\zeta^9]f_{n+\frac{13}{2}} \\
&+ [-\frac{13}{14h}\zeta^2 + \frac{677}{420h^2}\zeta^3 - \frac{11}{9h^3}\zeta^4 + \frac{1829}{3600h}{}^4\zeta^5 - \frac{107}{864h^5}\zeta^6 + \frac{89}{5040h} \\
&- \frac{11}{8064h^6}\zeta^7 + \frac{1}{22680h^7}\zeta^8 - \frac{1}{4536}h^8\zeta^9]f_{n+7}. \qquad (16)
\end{aligned}
$$

Evaluating (16) at$\zeta = 0, h, 2h, 3h, 4h, 5h, \frac{13}{2}$ and $7h$ yields the coefficients of the discrete HOBIM method k=7 in Table2.

Table 2: Coefficients of HOBIM $k = 7$

k	1	2	3	4	5	6	$\frac{13}{2}$	7
α_0	0	0	0	0	0	1	0	0
α_6	1	1	1	1	1	0	1	1
β_0	$\frac{241725}{31135104}$	$\frac{528}{467775}$	$\frac{2167}{3203200}$	$\frac{1353}{24324300}$	$\frac{332277}{778377600}$	$\frac{28523}{100100}$	$\frac{19090599}{19926466560}$	$\frac{115467}{778377600}$
β_1	$-\frac{1111685}{31135104}$	$-\frac{8328}{467775}$	$-\frac{24687}{3203200}$	$-\frac{1996}{24324300}$	$-\frac{332277}{778377600}$	$\frac{160992}{100100}$	$-\frac{187982769}{19926466560}$	$-\frac{1109667}{778377600}$
β_2	$-\frac{4075875}{31135104}$	$\frac{185306}{467775}$	$\frac{150579}{3203200}$	$\frac{8148291}{24324300}$	$\frac{15870569}{778377600}$	$-\frac{4719}{100100}$	$\frac{851282003}{19926466560}$	$\frac{4849559}{778377600}$
β_3	$-\frac{21503625}{31135104}$	$\frac{571560}{467775}$	$-\frac{1522235}{3203200}$	$-\frac{840840}{24324300}$	$-\frac{47775585}{778377600}$	$\frac{245960}{100100}$	$-\frac{2384941845}{19926466560}$	$-\frac{12833535}{778377600}$
β_4	$-\frac{39006825}{31135104}$	$\frac{376728}{467775}$	$-\frac{343500}{3203200}$	$\frac{10470603}{24324300}$	$\frac{112965567}{778377600}$	$-\frac{52767}{100100}$	$\frac{4822595349}{19926466560}$	$\frac{23326017}{778377600}$
β_5	$-\frac{27144975}{31135104}$	$\frac{5578832}{467775}$	$-\frac{3308877}{3203200}$	$\frac{28918032}{24324300}$	$-\frac{476877687}{778377600}$	$\frac{226512}{100100}$	$-\frac{8698280019}{19926466560}$	$-\frac{31923177}{778377600}$
β_6	$-\frac{22250085}{31135104}$	$\frac{217338}{467775}$	$-\frac{1853423}{3203200}$	$\frac{11821953}{24324300}$	$-\frac{482889693}{778377600}$	$-\frac{42757}{100100}$	$\frac{6050421649}{19926466560}$	$-\frac{84241443}{778377600}$
$\beta_{\frac{13}{2}}$	$\frac{6963200}{31135104}$	$-\frac{32768}{467775}$	$\frac{442368}{3203200}$	$-\frac{2080768}{24324300}$	$\frac{119554048}{778377600}$	$\frac{49152}{100100}$	$\frac{46852243456}{19926466560}$	$-\frac{556285952}{778377600}$
β_7	$-\frac{1104675}{31135104}$	$\frac{2904}{467775}$	$-\frac{60489}{3203200}$	$\frac{229944}{24324300}$	$-\frac{16195179}{778377600}$	$-\frac{10296}{100100}$	$-\frac{2146195623}{19926466560}$	$-\frac{120274869}{778377600}$

2.3. Derivation of HOBIM $k = 8$

Considering $k = 8$, $x \in [x_n, x_{n+8}]$ The continuous form of the method HOBIM k=8 is given by

$$y(x) = \sum_{j=0}^{k} \psi_j(x) y_{n+j} + h \sum_{j=0}^{k} \beta_j(x) f_{n+j} + \beta_j(\mu), \quad \mu = \frac{15}{2}. \tag{17}$$

Evaluating the matrix (8) with $s = 9, t = 2$ produces the matrix of the method D in (18)

$$D = \begin{pmatrix} 1 & x_{n+7} & x_{n+7}^2 & \dots & x_{n+7}^{10} \\ 0 & 1 & 2x_n & \dots & 10x_n^8 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2x_{n+\mu} & \dots & 9x_{n+\mu}^8 \\ 0 & 1 & 2x_{n+7} & \dots & 10x_{n+7}^9 \end{pmatrix} \tag{18}$$

Inverting (18) using the Maple software yields the elements of the matrix C .Substituting the elements of of C into (17) produces the continuous form of the multistep collocation method for k=8.which are substituted into (14) to give the continuous form of the multistep collocation method. Evaluating the method at$\zeta = 0, h, 2h, 3h, 4h, 5h, 6h\frac{15}{2} and 8h$ yields the coefficients of the discrete HOBIM method k=8 in Table3.

2.4. Derivation of HOBIM $k = 9$

Considering $k = 9$, $x \in [x_n, x_{n+9}]$ The continuous form of the method HOBIM k=9 is given by

$$y(x) = \sum_{j=0}^{k} \psi_j(x) y_{n+j} + h \sum_{j=0}^{k} \beta_j(x) f_{n+j} + \beta_j(\mu), \quad \mu = \frac{17}{2} \tag{19}$$

Evaluating the matrix (8) with $s = 10, t = 2$ produces the matrix of the method D in (20)

$$D = \begin{pmatrix} 1 & x_{n+8} & x_{n+8}^2 & \dots & x_{n+8}^{11} \\ 0 & 1 & 2x_n & \dots & 11x_n^{10} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2x_{n+\mu} & \dots & 11x_{n+\mu}^{10} \\ 0 & 1 & 2x_{n+9} & \dots & 11x_{n+9}^{10} \end{pmatrix} \tag{20}$$

Table 3: Coefficients of HOBIM k=8

k	1	2	3	4	5	6	7	$\frac{15}{2}$	8
α_0	0	0	0	0	0	0	1	0	0
α_7	1	1	1	1	1	1	0	1	1
β_0	$\frac{9}{140}$	$-\frac{12155}{10878368}$	$\frac{143}{779625}$	$-\frac{4719}{1601600}$	$\frac{121}{66237000}$	$-\frac{65923}{259459200}$	$\frac{738580}{18528000}$	$-\frac{6636839}{10218700800}$	$\frac{148577}{1297296000}$
β_1	$-\frac{482}{140}$	$-\frac{192775}{10878368}$	$-\frac{2200}{779625}$	$-\frac{54395}{1601600}$	$-\frac{70}{6237000}$	$\frac{719279}{259459200}$	$\frac{44021285}{185328000}$	$-\frac{71053180}{10218700800}$	$-\frac{1563925}{1297296000}$
β_2	$-\frac{1908}{140}$	$-\frac{4167995}{10878368}$	$\frac{19340}{779625}$	$-\frac{303615}{1601600}$	$-\frac{50}{6237000}$	$-\frac{3637699}{259459200}$	$-\frac{708625}{185328000}$	$\frac{349808000}{10218700800}$	$\frac{7522385}{1297296000}$
β_3	$-\frac{774}{140}$	$-\frac{12451725}{10878368}$	$-\frac{322190}{779625}$	$-\frac{1193335}{1601600}$	$\frac{26730}{6237000}$	$\frac{11443008}{259459200}$	$\frac{78863785}{185328000}$	$-\frac{1055993180}{10218700800}$	$-\frac{21949785}{1297296000}$
β_4	$-\frac{2090}{140}$	$-\frac{8991125}{10878368}$	$-\frac{931150}{779625}$	$-\frac{8319025}{1601600}$	$-\frac{19745}{6237000}$	$-\frac{25813645}{259459200}$	$-\frac{34809775}{185328000}$	$\frac{2218572950}{10218700800}$	$\frac{43675775}{1297296000}$
β_5	$-\frac{774}{140}$	$-\frac{10960235}{10878368}$	$-\frac{651904}{779625}$	$-\frac{16382223}{1601600}$	$\frac{2664442}{6237000}$	$\frac{48730253}{259459200}$	$\frac{79718639}{185328000}$	$-\frac{3588114772}{10218700800}$	$-\frac{63837631}{1297296000}$
β_6	$-\frac{1908}{140}$	$-\frac{10750025}{10878368}$	$-\frac{909700}{779625}$	$-\frac{17205045}{1601600}$	$\frac{7431820}{6237000}$	$-\frac{198188449}{259459200}$	$-\frac{844343}{185328000}$	$\frac{5389813880}{10218700800}$	$\frac{74006075}{1297296000}$
β_7	$-\frac{482}{140}$	$-\frac{598095}{10878368}$	$-\frac{379390}{779625}$	$-\frac{8700835}{1601600}$	$\frac{3016750}{6237000}$	$-\frac{1530552471}{259459200}$	$\frac{41487875}{185328000}$	$-\frac{31824380500}{10218700800}$	$\frac{122573165}{1297296000}$
$\beta_{\frac{15}{2}}$	0	$\frac{1392640}{10878368}$	$\frac{65536}{779625}$	$\frac{1851392}{1601600}$	$-\frac{524288}{6237000}$	$\frac{34816000}{259459200}$	$-\frac{19152896}{185328000}$	$-\frac{23519854592}{10218700800}$	$\frac{988491904}{1297296000}$
β_8	$-\frac{9}{140}$	0	$-\frac{6985}{779625}$	$-\frac{231660}{1601600}$	$\frac{57145}{6237000}$	$-\frac{4409548}{259459200}$	$\frac{3343340}{185328000}$	$\frac{1001060555}{10218700800}$	$\frac{198229460}{1297296000}$

Inverting (20) using the Maple software yields the elements of the matrix C.Substituting the elements of the continuous coefficients into (19) produces the continuous form of the multistep collocation method for $k = 9$. Evaluating the continuous form of the linear multi-step method at $\zeta = x - x_n, \zeta = 0, h, 2h, 3h, 4h, 5h, 6h, 7h, \frac{17}{2}$ and $9h$ produces the coefficients of the discrete HOBIM method for $k = 9$ in $Table4$ used as block integrators.

2.5. Derivation of HOBIM $k = 10$

Considering $k = 10$, $x \in [x_n, x_{n+10}]$ The continuous form of the method HOBIM $k = 10$ is given by

$$y(x) = \sum_{j=0}^{k} \psi_j(x) y_{n+j} + h \sum_{j=0}^{k} \beta_j(x) f_{n+j} + \beta_j(\mu), \quad \mu = \frac{19}{2} \tag{21}$$

Evaluating the matrix (8) with $s = 10, t = 2$ produces the matrix of the method D in (22)

$$D = \begin{pmatrix} 1 & x_{n+9} & x_{n+9}^2 & \dots & x_{n+9}^{12} \\ 0 & 1 & 2x_n & \dots & 12x_n^{11} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 2x_{n+\mu} & \dots & 12x_{n+\mu}^{11} \\ 0 & 1 & 2x_{n+9} & \dots & 12x_{n+9}^{11} \end{pmatrix} \tag{22}$$

Inverting [22] using the Maple software yields the elements of the matrix (9).Substituting the elements of the continuous corfficients into(21) produces the continuous form of the multistep collocation method for $k = 10$.Evaluating the continuous form of the linear multi-step method at $\zeta = x - x_n, \zeta = 0, h, 2h, 3h, 4h, 5h, 6h, 7h, 8h, \frac{19}{2}$ and $10h$ produces the coefficients of the discrete HOBIM method for $k = 10$ in Table 5 used as block integrators.

3. Analysis of the New Methods

In this section, we determine the convergence, construct the regions of absolute stability and obtain the orders of the new HOBIM methods.

3.1. Convergence Analysis

Using the approach of Fatunla [15],[16] we determine the convergence of the new HOBIM methods where the block methods are represented as a single block, r

Table 4: Coefficients of HOBIM $k = 9$

k	1	2	3	4	5	6	7	8	$\frac{17}{2}$	9
α_0	0	0	0	0	0	0	0	1	0	0
α_8	1	1	1	1	1	1	0	1	1	1
β_0	$\frac{487806319}{83175206400}$	$-\frac{27170}{37437400}$	$\frac{740882}{23289057792}$	$-\frac{6721}{1137161025}$	$\frac{374803}{2395993600}$	$-\frac{322322}{9097288200}$	$\frac{94307785}{582226444800}$	$\frac{310589708}{1137161025}$	$\frac{13779794171}{2980999939737600}$	$\frac{52865527}{582226444800}$
β_1	$-\frac{28178875725}{83175206400}$	$\frac{517808}{37437400}$	$-\frac{102794835}{23289057792}$	$-\frac{277134}{1137161025}$	$-\frac{4528953}{2395993600}$	$\frac{3464175}{9097288200}$	$-\frac{1109870619}{582226444800}$	$\frac{1958491392}{1137161025}$	$-\frac{159914887275}{2980999939737600}$	$\frac{605909733}{582226444800}$
β_2	$-\frac{11533848400}{83175206400}$	$-\frac{14168242}{37437400}$	$-\frac{781136400}{23289057792}$	$\frac{3525324}{1137161025}$	$\frac{25782064}{2395993600}$	$-\frac{16297050}{9097288200}$	$\frac{6053782416}{582226444800}$	$-\frac{616615296}{1137161025}$	$\frac{855986646900}{2980999939737600}$	$-\frac{319063588}{582226444800}$
β_3	$-\frac{41062788312}{83175206400}$	$-\frac{47494668}{37437400}$	$-\frac{10314131880}{23289057792}$	$-\frac{29081832}{1137161025}$	$-\frac{94077048}{2395993600}$	$\frac{41113956}{9097288200}$	$-\frac{20405861736}{582226444800}$	$\frac{4247273472}{1137161025}$	$-\frac{2809327722708}{2980999939737600}$	$\frac{810238952984}{582226444800}$
β_4	$-\frac{131660563034}{83175206400}$	$-\frac{27163994}{37437400}$	$-\frac{26179560550}{23289057792}$	$\frac{471545932}{1137161025}$	$\frac{267726030}{2395993600}$	$-\frac{37724258}{9097288200}$	$\frac{48123351562}{582226444800}$	$-\frac{3068699920}{1137161025}$	$\frac{6360134317394}{2980999939737600}$	$-\frac{22433058934}{582226444800}$
β_5	$-\frac{39910344474}{83175206400}$	$-\frac{47574670}{37437400}$	$-\frac{22441654950}{23289057792}$	$\frac{1356118764}{1137161025}$	$-\frac{1359215858}{2395993600}$	$-\frac{189370038}{9097288200}$	$-\frac{86784001590}{582226444800}$	$\frac{5440461312}{1137161025}$	$-\frac{10688638026066}{2980999939737600}$	$\frac{35778539082}{582226444800}$
β_6	$-\frac{111126591576}{83175206400}$	$-\frac{28457286}{37437400}$	$-\frac{22480234920}{23289057792}$	$\frac{952786692}{1137161025}$	$-\frac{2343736824}{2395993600}$	$\frac{3794299938}{9097288200}$	$\frac{136284991128}{582226444800}$	$-\frac{2232008064}{1137161025}$	$\frac{14402757628116}{2980999939737600}$	$-\frac{43748723304}{582226444800}$
β_7	$-\frac{72986593680}{83175206400}$	$-\frac{45479148}{37437400}$	$-\frac{25633436400}{23289057792}$	$\frac{1325517336}{1137161025}$	$-\frac{2650334544}{2395993600}$	$\frac{10905806340}{9097288200}$	$-\frac{396653667696}{582226444800}$	$\frac{3270998016}{1137161025}$	$-\frac{18534243061620}{2980999939737600}$	$\frac{43998493968}{582226444800}$
β_8	$-\frac{55703941293}{83175206400}$	$-\frac{17046172}{37437400}$	$-\frac{12237179955}{23289057792}$	$\frac{55440656}{1137161025}$	$-\frac{124300067}{2395993600}$	$\frac{435091284}{9097288200}$	$-\frac{329020653819}{582226444800}$	$-\frac{718827252}{1137161025}$	$\frac{94946391302763}{2980999939737600}$	$\frac{46922410821}{582226444800}$
$\beta_{\frac{17}{2}}$	$\frac{154559024896}{83175206400}$	$\frac{2490368}{37437400}$	$\frac{2446458880}{23289057792}$	$-\frac{96206848}{1137161025}$	$\frac{242614272}{2395993600}$	$-\frac{735182848}{9097288200}$	$\frac{69482971136}{582226444800}$	$\frac{620756992}{1137161025}$	$\frac{6734908122764}{2980999939737600}$	$\frac{426040754176}{582226444800}$
β_9	$-\frac{2210729521}{83175206400}$	$-\frac{221221}{37437400}$	$-\frac{291294575}{23289057792}$	$\frac{10302578}{1137161025}$	$-\frac{28284685}{2395993600}$	$\frac{77872223}{9097288200}$	$-\frac{8291793367}{582226444800}$	$-\frac{1155132160}{1137161025}$	$-\frac{2686037350139}{2980999939737600}$	$\frac{88066667689}{582226444800}$

Table 5: Coefficients of HOBIM $k = 10$

k	1	2	3	4	5	6	7	8	9	$\frac{19}{2}$	10
α_0	0	0	0	0	0	0	0	0	1	0	0
α_9	1	1	1	1	1	1	1	1	1	1	1
β_0	$\frac{2368}{467775}$	$\frac{104261612}{143666266600}$	$\frac{663}{6806800}$	$\frac{5220020}{40226554368}$	$\frac{58123}{1964187225}$	$\frac{391612}{4138534400}$	$-\frac{58123}{1654052400}$	$-\frac{109454228}{1005663859200}$	$\frac{1110235932}{4138534400}$	$-\frac{8279227567}{242305833369600}$	$\frac{74000524}{100663859200}$
β_1	$-\frac{154228}{467775}$	$-\frac{2007851391}{143666266600}$	$-\frac{12350}{6806800}$	$-\frac{71129825}{40226554368}$	$-\frac{638001}{1964187225}$	$-\frac{5024227}{4138534400}$	$\frac{708214}{1654052400}$	$\frac{1386207225}{1005663859200}$	$\frac{7351165341}{4138534400}$	$\frac{108733231706}{242305833369600}$	$-\frac{918377863}{100663859200}$
β_2	$-\frac{670656}{467775}$	$\frac{5461551540}{143666266600}$	$\frac{13193}{6806800}$	$\frac{475095855}{40226554368}$	$\frac{2834325}{1964187225}$	$\frac{3014462}{4138534400}$	$-\frac{3942861}{1654052400}$	$-\frac{8155940247}{1005663859200}$	$-\frac{3375681075}{4138534400}$	$-\frac{601900264719}{242305833369600}$	$\frac{5264601225}{100663859200}$
β_3	$-\frac{161664}{467775}$	$\frac{180925400880}{143666266600}$	$-\frac{2692324}{6806800}$	$-\frac{2264746800}{40226554368}$	$-\frac{4104684}{1964187225}$	$-\frac{113215376}{4138534400}$	$\frac{13156980}{1654052400}$	$\frac{29653120944}{1005663859200}$	$\frac{18904197744}{4138534400}$	$\frac{2147824384920}{242305833369600}$	$-\frac{18489651792}{100663859200}$
β_4	$-\frac{888192}{467775}$	$\frac{10874516452}{143666266600}$	$-\frac{8394906}{6806800}$	$\frac{19708762320}{40226554368}$	$-\frac{29149458}{1964187225}$	$\frac{30454507}{4138534400}$	$-\frac{28558062}{1654052400}$	$-\frac{74948522064}{1005663859200}$	$-\frac{18192285072}{4138534400}$	$-\frac{5286619356798}{242305833369600}$	$\frac{44527472496}{100663859200}$
β_5	$\frac{254}{467775}$	$\frac{171946522202}{143666266600}$	$-\frac{5291624}{6806800}$	$\frac{42442190310}{40226554368}$	$\frac{78356699}{1964187225}$	$-\frac{670773454}{4138534400}$	$\frac{37780392}{1654052400}$	$\frac{141351801994}{1005663859200}$	$\frac{29860860978}{4138534400}$	$\frac{9574283911868}{242305833369600}$	$-\frac{78116289654}{100663859200}$
β_6	$-\frac{888192}{467775}$	$\frac{128365168386}{143666266600}$	$-\frac{8272030}{6806800}$	$\frac{4173826995}{40226554368}$	$\frac{237551706}{1964187225}$	$\frac{2570955322}{4138534400}$	$\frac{1300806}{1654052400}$	$-\frac{212288549070}{1005663859200}$	$-\frac{18436944006}{4138534400}$	$-\frac{13407250515126}{242305833369600}$	$\frac{103979593458}{100663859200}$
β_7	$-\frac{161664}{467775}$	$\frac{144402635832}{143666266600}$	$-\frac{5476380}{6806800}$	$\frac{36449167560}{40226554368}$	$\frac{161921838}{1964187225}$	$\frac{3869697624}{4138534400}$	$-\frac{663368628}{1654052400}$	$\frac{285312476280}{1005663859200}$	$\frac{19277944920}{4138534400}$	$\frac{15482382099288}{242305833369600}$	$-\frac{109310215560}{100663859200}$
β_8	$-\frac{670656}{467775}$	$\frac{155782312140}{143666266600}$	$-\frac{8079981}{6806800}$	$\frac{45763641300}{40226554368}$	$\frac{2306094999}{1964187225}$	$\frac{4689459996}{4138534400}$	$-\frac{1999438935}{1654052400}$	$-\frac{716323517364}{1005663859200}$	$-\frac{3753553284}{4138534400}$	$-\frac{17424846944115}{242305833369600}$	$\frac{97087548012}{100663859200}$
β_9	$\frac{154228}{467775}$	$\frac{76406192317}{143666266600}$	$-\frac{3225274}{6806800}$	$\frac{2014514339}{40226554368}$	$\frac{946567973}{1964187225}$	$\frac{2074839273}{4138534400}$	$-\frac{780032318}{1654052400}$	$-\frac{547512098731}{1005663859200}$	$\frac{7071716457}{4138534400}$	$\frac{78748729590778}{242305833369600}$	$\frac{66987700949}{100663859200}$
$\beta_{\frac{19}{2}}$	0	$-\frac{15459024896}{143666266600}$	$\frac{524288}{6806800}$	$-\frac{3662807040}{40226554368}$	$-\frac{159907840}{1964187225}$	$-\frac{376832000}{4138534400}$	$\frac{127401984}{1654052400}$	$\frac{108213174272}{1005663859200}$	$-\frac{3061186560}{4138534400}$	$\frac{538550857537536}{242305833369600}$	$\frac{743868334080}{100663859200}$
β_{10}	0	$\frac{183756218}{143666266600}$	$-\frac{52819}{6806800}$	$\frac{403964795}{40226554368}$	$\frac{1669102}{1964187225}$	$\frac{41414737}{4138534400}$	$-\frac{13054249}{1654052400}$	$-\frac{12242558211}{1005663859200}$	$\frac{490338225}{4138534400}$	$-\frac{2025997762971}{242305833369600}$	$\frac{1507122610825}{100663859200}$

point multi-step method of the form

$$A^0 y_{m+1} = \sum_{i=1}^{k} A^i y_{m+1} + h \sum_{j=0}^{k} B^i f_{m-1} \tag{23}$$

h being a fixed mesh size within a block, $A^i, B^i, i = 0(1)k$ are rxr identity matrix while Y_m, Y_{m-1}andF_{m-1} are vectors of numerical estimates.

Definition 1. Zero stability For $n = mr$,for some integer $m \geq 0$, the block method (23) is zero stable if the roots $R_j, N = 1(1)k$ of the first characteristic polynomial $\rho(R)$ given by

$$\rho(R) = det[\sum_{i=0}^{k} A^i R^i] = 0 \tag{24}$$

satisfies $R_j \leq 1$ and for those roots with $R_j \leq 1$,the multiplicity nust not exceed two.The block method with coefficients in Table 1 is expressed in the form of (23) gives

$$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_{n+1} \\ y_{n+2} \\ y_{n+3} \\ y_{n+4} \\ y_{n+5} \\ y_{n+6} \\ y_{n+\frac{11}{2}} \\ y_{n+7} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_{n-6} \\ y_{n-5} \\ y_{n-4} \\ y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix}$$

$$+h \begin{pmatrix} -\frac{342}{945} & -\frac{1224}{945} & -\frac{664}{945} & -\frac{1224}{945} & -\frac{342}{945} & 0 & \frac{8}{945} \\ \frac{979}{36960} & -\frac{15030}{36960} & -\frac{41910}{36960} & -\frac{35960}{36960} & -\frac{23661}{36960} & \frac{6656}{36960} & -\frac{1023}{36960} \\ -\frac{550}{124740} & \frac{3861}{124740} & -\frac{53064}{124740} & -\frac{148929}{124740} & -\frac{59994}{124740} & \frac{10240}{124740} & -\frac{1089}{124740} \\ \frac{2475}{332640} & -\frac{11187}{332640} & \frac{35354}{332640} & -\frac{190872}{332640} & -\frac{219285}{332640} & \frac{59904}{332640} & -\frac{8767}{332640} \\ \frac{28025}{18144} & \frac{3375}{18144} & \frac{35550}{18144} & \frac{3000}{18144} & \frac{25785}{18144} & -\frac{12800}{18144} & \frac{2475}{18144} \\ \frac{21150}{15482880} & -\frac{88893}{15482880} & \frac{23972}{15482880} & -\frac{540873}{15482880} & \frac{4566222}{15482880} & \frac{3732450}{15482880} & -\frac{186043}{15482880} \\ -\frac{157}{90720} & \frac{621}{90720} & -\frac{1494}{90720} & \frac{2496}{90720} & \frac{11043}{90720} & \frac{64000}{90720} & \frac{14193}{90720} \end{pmatrix}$$

$$\times \begin{pmatrix} f_{n+1} \\ f_{n+2} \\ f_{n+3} \\ f_{n+4} \\ f_{n+5} \\ f_{n+\frac{11}{2}} \\ f_{n+6} \end{pmatrix} + h \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{8}{945} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{78}{36960} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{45}{124740} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{262}{332640} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1062}{18144} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{2335}{15482880} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{18}{90720} \end{pmatrix} \begin{pmatrix} f_{n-6} \\ f_{n-5} \\ f_{n-4} \\ f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_{n} \end{pmatrix}, \quad (25)$$

where

$$A^0 = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}, \quad (26)$$

$$A^1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$B^0 =$$
$$\begin{pmatrix} -\frac{342}{945} & -\frac{1224}{945} & -\frac{664}{945} & -\frac{1224}{945} & -\frac{342}{945} & 0 & \frac{8}{945} \\ \frac{979}{36960} & -\frac{15030}{36960} & -\frac{41910}{36960} & -\frac{35960}{36960} & -\frac{23661}{36960} & \frac{6656}{36960} & -\frac{1023}{36960} \\ -\frac{550}{124740} & \frac{3861}{124740} & -\frac{53064}{124740} & -\frac{148929}{124740} & -\frac{59994}{124740} & \frac{10240}{124740} & -\frac{1089}{124740} \\ \frac{2475}{332640} & -\frac{11187}{332640} & \frac{35354}{332640} & -\frac{190872}{332640} & -\frac{219285}{332640} & \frac{59904}{332640} & -\frac{8767}{332640} \\ \frac{28025}{18144} & \frac{3375}{18144} & \frac{35550}{18144} & \frac{3000}{18144} & \frac{25785}{18144} & -\frac{12800}{18144} & \frac{2475}{18144} \\ \frac{21150}{15482880} & -\frac{88893}{15482880} & \frac{23972}{15482880} & -\frac{540873}{15482880} & \frac{4566222}{15482880} & \frac{3732450}{15482880} & -\frac{186043}{15482880} \\ -\frac{157}{90720} & \frac{621}{90720} & -\frac{1494}{90720} & \frac{2496}{90720} & \frac{11043}{90720} & \frac{64000}{90720} & \frac{14193}{90720} \end{pmatrix},$$

$$B^1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{8}{945} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{78}{36960} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{45}{124740} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{262}{332640} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1062}{18144} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{2335}{15482880} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{18}{90720} \end{pmatrix}$$

Substituting (26) into (24) produces the zero stability polynomial on a parameter R below:

$$\rho(R) = det \left[R \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right]$$

$$= det \begin{bmatrix} R & 0 & 0 & 0 & -R & 0 & 0 \\ 0 & R & 0 & 0 & -R & 0 & 0 \\ 0 & 0 & R & 0 & -R & 0 & 0 \\ 0 & 0 & 0 & R & -R & 0 & 0 \\ 0 & 0 & 0 & 0 & R & 0 & -1 \\ 0 & 0 & 0 & 0 & -R & R & 0 \\ 0 & 0 & 0 & 0 & -R & 0 & R \end{bmatrix}$$

$$= R^6(R(R-1)) = 0$$

$$\Rightarrow R_1 = 1, R_2 = R_3 = R_4 = R_5 = R_6 = 0$$

By definition (1),the block method whose coefficients are in Table I is zero stable and of order $p \geq 1$.Therefore, by Henrichi [21],it is convergent. Using a similar approach, the block methods whose coefficients are in Tables (2,3,4,and 5) are also convergent.

3.2. Order of the Methods

Using the method in Chollom, et al[11] the new HOBIM methods have orders and error constants as shown in Table.6 below:

3.3. Regions of Absolute Stability of the HOBIM Methods

The absolute stability regions of the HOBIM methods are constructed by reformulating the block integrators whose coefficients are in Tables 1-5 as General linear Methods of Butchers [4] using the notations introduced in Burage and Butchers [5] .The General linear method (GLM) is represented by a partitioned (s+r) $\times$ (s+r) characterized by the four matrices A, B, U and V expressed in

Table 6: Order and error constants of the HOBIM methods

method	order	error constant	method	order	error constant
HOBIM k=6, $\mu = \frac{11}{2}$	8	$-\frac{13}{14175}$ $\frac{197}{358400}$ $\frac{10069}{29030400}$ $-\frac{7325}{1161216}$ $\frac{41}{907200}$ $\frac{578503}{741782400}$ $-\frac{3499}{29030400}$ 0	HOBIM k=7, $\mu = \frac{13}{2}$	9	$\frac{608349}{11147673600}$ $-\frac{1039}{10886400}$ $\frac{13}{85050}$ $\frac{11}{44800}$ $\frac{425}{435456}$ $\frac{11}{640800}$ $\frac{461}{2177280}$ $\frac{461}{2177280}$
HOBIM k=8, $\mu = \frac{15}{2}$	10	$\frac{19}{30800}$ $\frac{10355}{38320128}$ $-\frac{47}{9355500}$ $\frac{2621}{19712000}$ $\frac{161}{5346000}$ $\frac{131899}{958003200}$ $\frac{3411233}{684288000}$ $\frac{96362197}{2452488192000}$ $\frac{369689}{4790016000}$	HOBIM k=9, $\mu = \frac{17}{2}$	11	$\frac{117943}{188179200}$ $\frac{57}{677600}$ $\frac{5905}{52690176}$ $\frac{263}{10291050}$ $\frac{443}{5420800}$ $\frac{4997}{164656800}$ $\frac{123817}{1317254400}$ $\frac{636863659}{21581896089600}$
HOBIM k=10, $\mu = \frac{19}{2}$	12	$\frac{275216}{638512875}$ $\frac{31728431}{213497856000}$ $\frac{1613}{336336000}$ $\frac{5124295}{83691159552}$ $\frac{148111}{510810300}$ $\frac{2354117}{430510008000}$ $\frac{697840127}{10461394944000}$ $\frac{58336407}{14350336000}$ $\frac{488019910837}{21424936845312000}$ $\frac{555959297}{10461394944000}$			

the form .

$$\begin{bmatrix} Y^1_{[n]} \\ \vdots \\ Y^i_{[n]} \\ \hline Y^1_{[n]} \\ \vdots \\ Y^r_{[n]} \end{bmatrix} = \left[\begin{array}{c|c} A & B \\ \hline U & V \end{array}\right] \begin{bmatrix} hf(Y^1_{[n]}) \\ \vdots \\ hf(Y^i_{[n]}) \\ \hline y^1_{[n-1]}) \\ \vdots \\ y^r_{[n-1]}) \end{bmatrix}$$

The elements of these matrices A,B,U and V are substituted into the recurrence relation

$$y^{[i-1]} = M(z)y^{[i]}, i = 1, 2, \ldots, N-1$$

where

$$M(z) = U + zB(I - zA)^{-1}V$$

The stability polynomial

$$\rho(\eta, z) = \det(\eta I - M(z))$$

of the method plotted in a matlab environment produces the regions of absolute stability of the HOBIM methods in figures a-e.

4. Numerical Examples

To check the behavior of the new HOBIM integrators on Stiff Differential Equations,we solve well known numerical problems using a fix dtep size. The test problems are solved and results obtained are compared with those using ode 23s or exact solutions to illustrate their potential.

Example 4.1 Stiff linear system

$$y' = \begin{pmatrix} -2 & 1 \\ 998 & -999 \end{pmatrix} y + \begin{pmatrix} 2sinx \\ 999(cosx - sinx) \end{pmatrix}, y(0) = \begin{pmatrix} 2 \\ 3 \end{pmatrix},$$
$$0 \leq x \leq 100, h = 0.1$$

Example 4.2 Robertson equation

$$\begin{array}{l} y_1' = -0.04y_1 + 10000y_2y_3 \\ y_2' = 0.04y_1 - 10000y_2y_3 - 3.0 \times 10^7 y_2^2 \\ y_3' = 3.0 \times 10^7 y_2^2 \end{array}, y(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, 0 \leq x \leq 40, h = 0.1$$

Example 4.3 Chemical reaction equation

$$y' = \begin{pmatrix} -500000.5 & 499999.5 \\ 499999.5 & -500000.5 \end{pmatrix} y, y(0) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, 0 \leq x \leq 40, h = 0.1$$

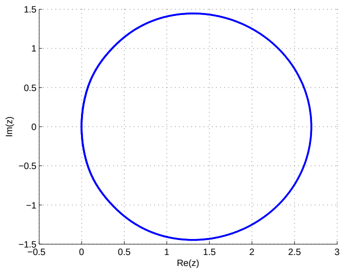

Figure 1: Absolute stability region of HOBIM k=6

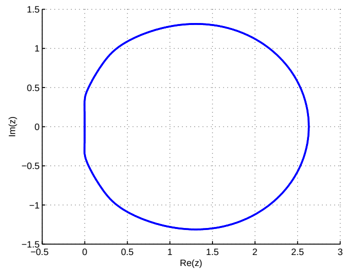

Figure 2: Absolute stability region of HOBIM k=7.

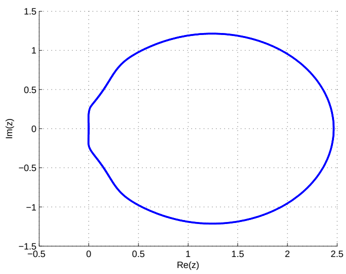

Figure 3: Absolute stability region of HOBIM k=8

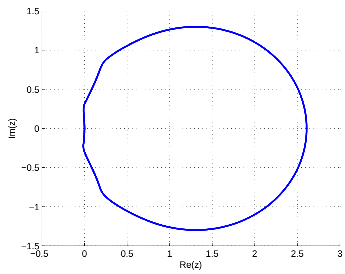

Figure 4: Absolute stability region of HOBIM k=9.

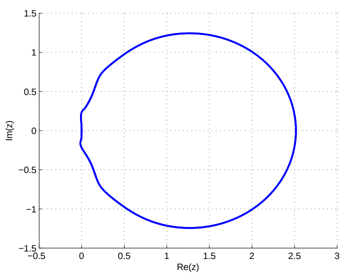

Figure 5: Absolute stability region of HOBIM k=10.

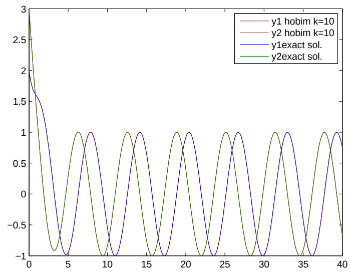

Figure 6: Solution curve of example 4.1

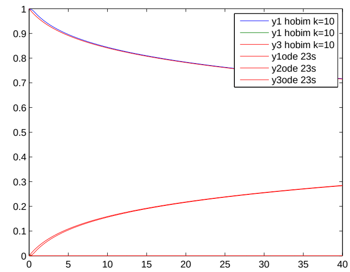

Figure 7: Solution curve of example 4.2

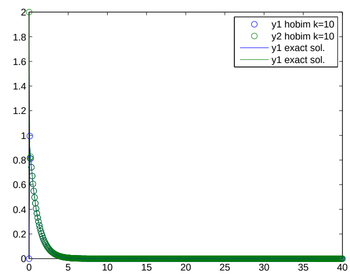

Figure 8: Solution curve of example 4.3

5. Concluding Remarks

The search for high order accurate A-stable numerical methods for Stiff Ordinary Differential equations has been pursued since 1952 by various researchers. We have in this paper constructed High Order Block Implicit multi-step (HOBIM) methods for a class of the Adams Moultons family using the multi-step collocation approach. These members are of high order and their regions of absolute stability plotted in figures 1-6 are shown to be A-stable, a property desirable of methods required for the solution of Stiff Ordinary Differential Equations. The new HOBIM member k=10 is tested on the Robertson equation problem 1 used in Aguiar and Ramos[1], a stiff linear equation problem 2 used in Kumleng[22] and a chemical reaction equation problem 3 used in Tahmasbi [29].The result of problem 1 using the HOBIM member k=10 is compared to the results obtained using the inbuilt matlab solver ode23s, the problem 2,stiff linear equation and the chemical reaction equation in problem 3 solved using the HOBIM method had their results compared to that of the exact solutions. The solution curves displayed in figures 6, 7 and 8 respectively show the efficiency and suitability of the HOBIM methods for the solution of Stiff Ordinary Differential Equations.

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